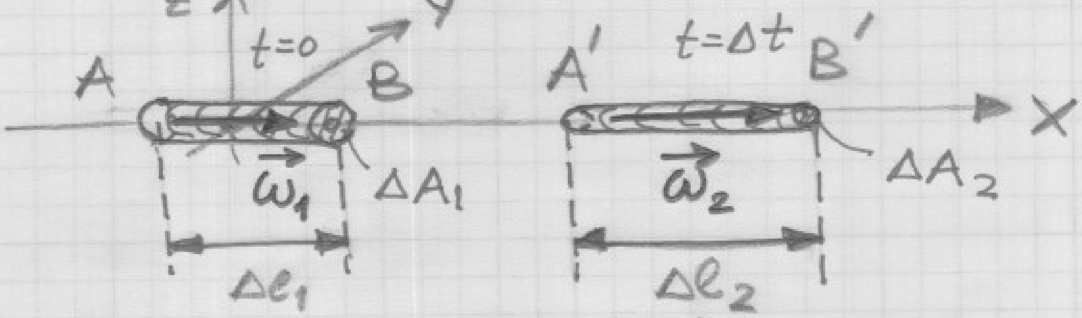
Physical meaning of vortex stretching term in the vorticity equation

Vorticity equ:

$$\frac{D\vec{\omega}}{Dt} = \underbrace{(\vec{\omega}\cdot\vec{\nabla})\vec{q}}_{A} + \underbrace{\nu\nabla^2\vec{\omega}}_{B} \quad (1)$$

According to equ. (1) the vorticity, $\vec{\omega}$, of fluid particles changes, as they move with the flow (that is what $\frac{D}{Dt}$ is!), and that rate of change with time has two components: parts A and B. B corresponds to viscous diffusion and A to vortex stretching. In order to see the physical meaning of term A we will apply $\frac{D\vec{\omega}}{Dt} = (\vec{\omega}\cdot\vec{\nabla})\vec{q}$ in a particular case:



Vortex element AB (at t=0) moves to A'B' (at t=Δt)
↳ defined as a thin tube inside which $\vec{\omega}_1$ = const ≠ 0
(Note $\vec{\omega} = 0$ outside AB)

STAEDTLER® No. 937 811E Engineer's Computation Pad

First of all conservation of mass would require: $\underline{\Delta l_1 \cdot \Delta A_1 = \Delta l_2 \cdot \Delta A_2}$ (2)

(ΔA_1, ΔA_2 are crossectional areas)

We will first see how the $\vec{\omega}\cdot\vec{\nabla}$ operator becomes in this case:

$$\vec{\omega}_1 \vec{\nabla} = \underbrace{\omega_{1x}}_{=|\vec{\omega}_1| = \omega_1} \frac{\partial}{\partial x} + \overset{=0}{\cancel{\omega_{1y}}} \frac{\partial}{\partial y} + \overset{=0}{\cancel{\omega_{1z}}} \frac{\partial}{\partial z}$$

Thus $\vec{\omega}_1 \vec{\nabla} \equiv \omega_1 \frac{\partial}{\partial x}$

eq.(3)

Then: $$(\vec{\omega}\cdot\vec{\nabla})\vec{q} = \omega_1 \frac{\partial u}{\partial x}\vec{i} + \omega_1 \frac{\partial v}{\partial x}\vec{j} + \omega_1 \frac{\partial w}{\partial x}\vec{k}$$

(where $\vec{q} = u\vec{i} + v\vec{j} + w\vec{k}$, velocity vector)

We will then consider only the (x) component of eq. (3), and we will equate it to the x component of the LHS of equ. (1) [remember we do not consider the B term here!]

STAEDTLER® No. 937 811E Engineer's Computation Pad

$$\frac{D\omega_{1x}}{Dt} = \omega_1 \frac{\partial u}{\partial x} \qquad (4)$$

We will now express the 2 sides of eq. (4) in terms of discrete differences (which should be "exact" as $\Delta t \to 0$)

$$\frac{D\omega_{1x}}{Dt} \approx \frac{\omega_2 - \omega_1}{\Delta t} \quad (5) \quad \left(\omega_2 = |\vec{\omega}_2|\right)$$

$$\frac{\partial u}{\partial x} \approx \frac{u_B - u_A}{\Delta l_1}$$

But: $\left(u_B - u_A\right)\Delta t = \Delta l_2 - \Delta l_1$

Thus $\quad \dfrac{\partial u}{\partial x} \approx \dfrac{\Delta l_2 - \Delta l_1}{\Delta t \cdot \Delta l_1} \quad (6)$

Replacing (5) & (6) into (4) we get:

$$\frac{\omega_2 - \omega_1}{\cancel{\Delta t}} = \omega_1 \frac{\Delta l_2 - \Delta l_1}{\cancel{\Delta t} \cdot \Delta l_1} \rightsquigarrow$$

$$\rightsquigarrow \quad \omega_2 - \omega_1 = \omega_1 \left(\frac{\Delta l_2}{\Delta l_1} - 1\right) \rightsquigarrow \boxed{\frac{\omega_2}{\omega_1} = \frac{\Delta l_2}{\Delta l_1}} \quad (7)$$

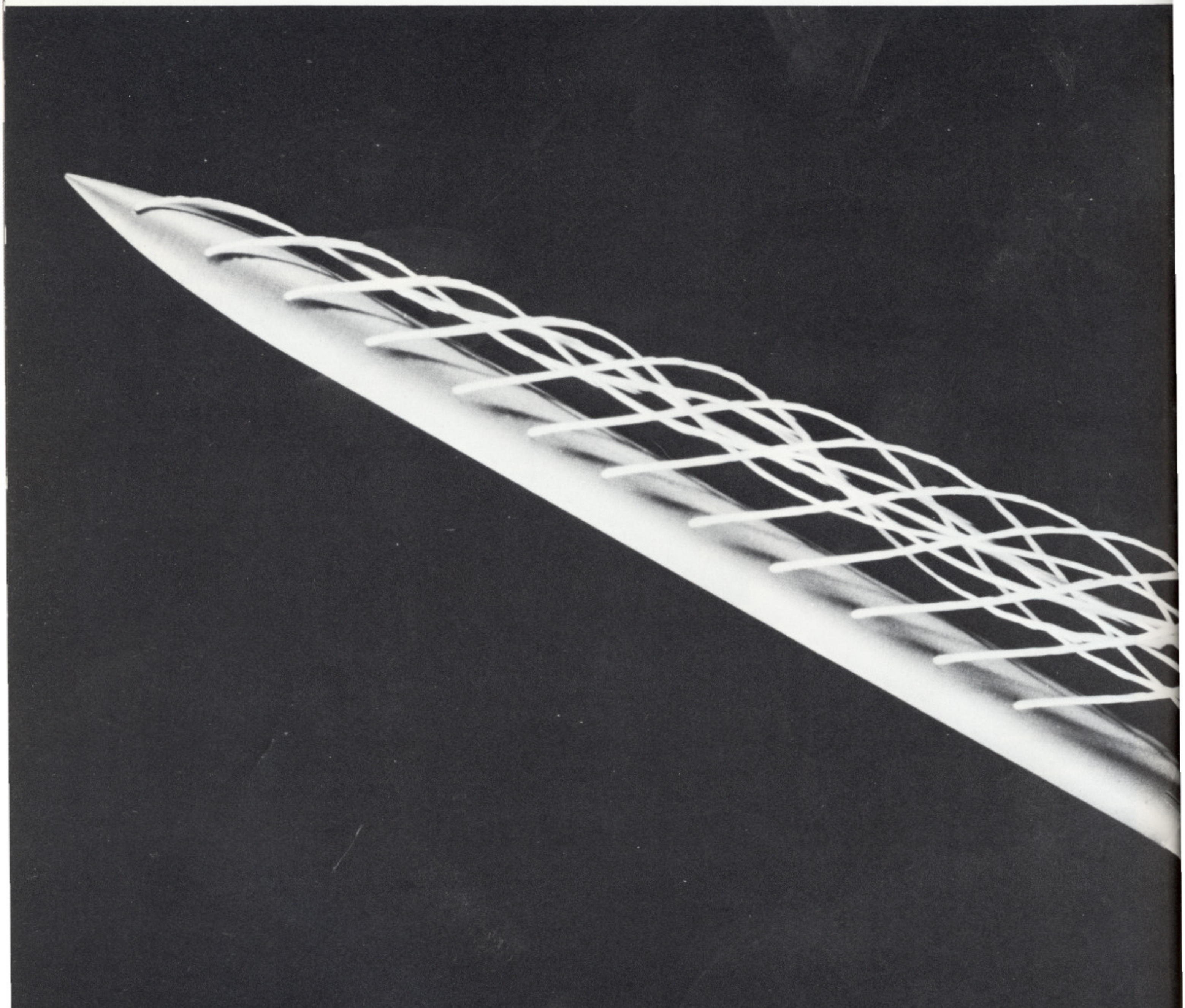
Eq. (7) states that as the vortex element stretches its vorticity (which is directly related to the spin of the particles) <u>increases</u> By using (2) eq. (7) can also be written as: $\boxed{\omega_2 \Delta A_2 = \omega_1 \Delta A_1}$ (8)

As we explained in class eq. (7) or eq. (8) make physical sense since a stretched vortex gets "tighter" (i.e. ΔA decreases), and for angular momentum to be conserved (remember classic demo where a spinning person rotates faster as they move their arms towards their body), ω must increase.

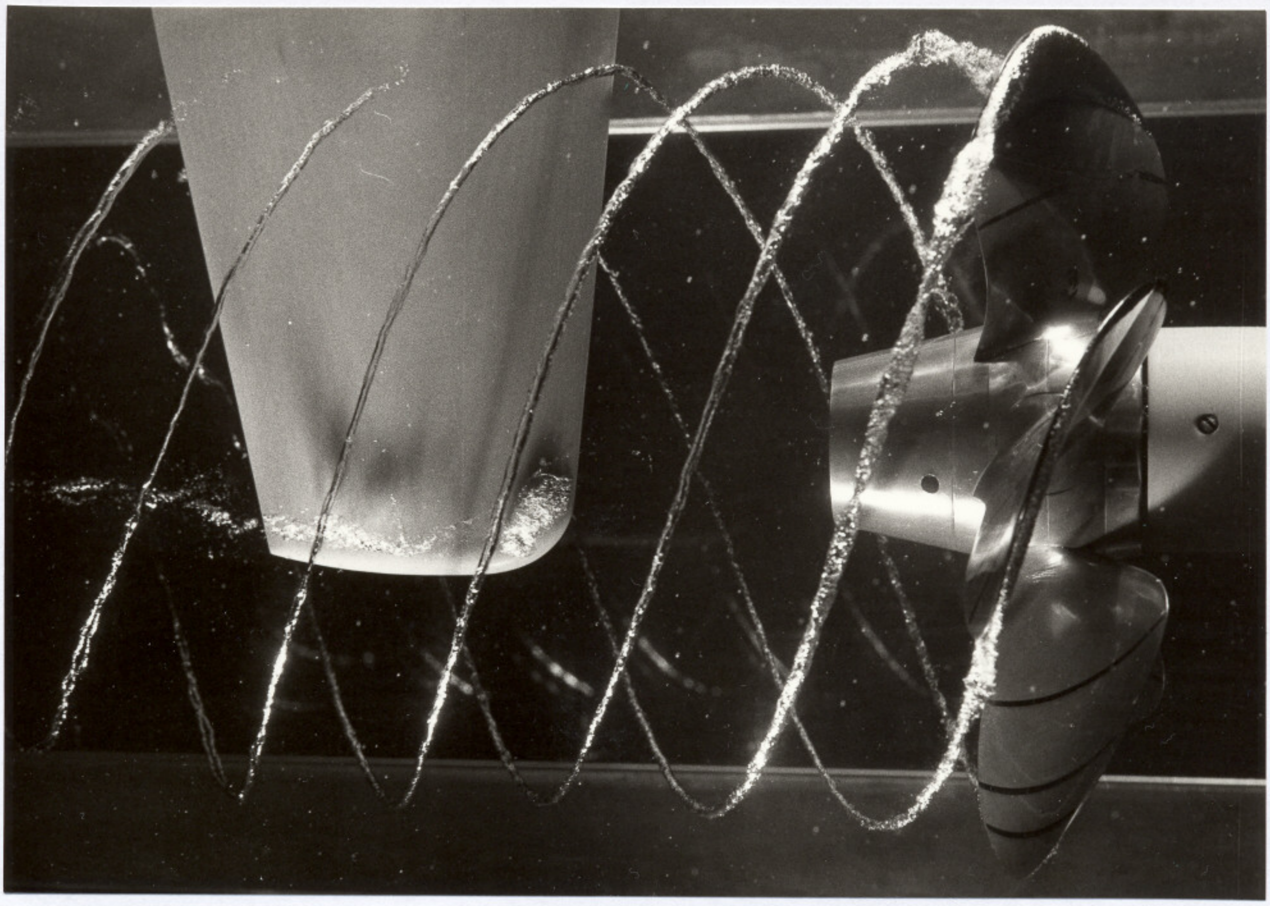
In the case of a vortex element we often define: $\vec{\Gamma} = \vec{\omega}\,\Delta A$; where $\vec{\Gamma}$ is now called a <u>discrete vortex</u> and $\underline{\Gamma = \omega\,\Delta A}$ is the <u>vortex strength</u>

So according to (8): $\boxed{\Gamma_2 = \Gamma_1}$ or the vortex strength is <u>conserved</u> (again in the absence of term B in (1))

$\Rightarrow$ See next 2 pages for nice photos of vortices



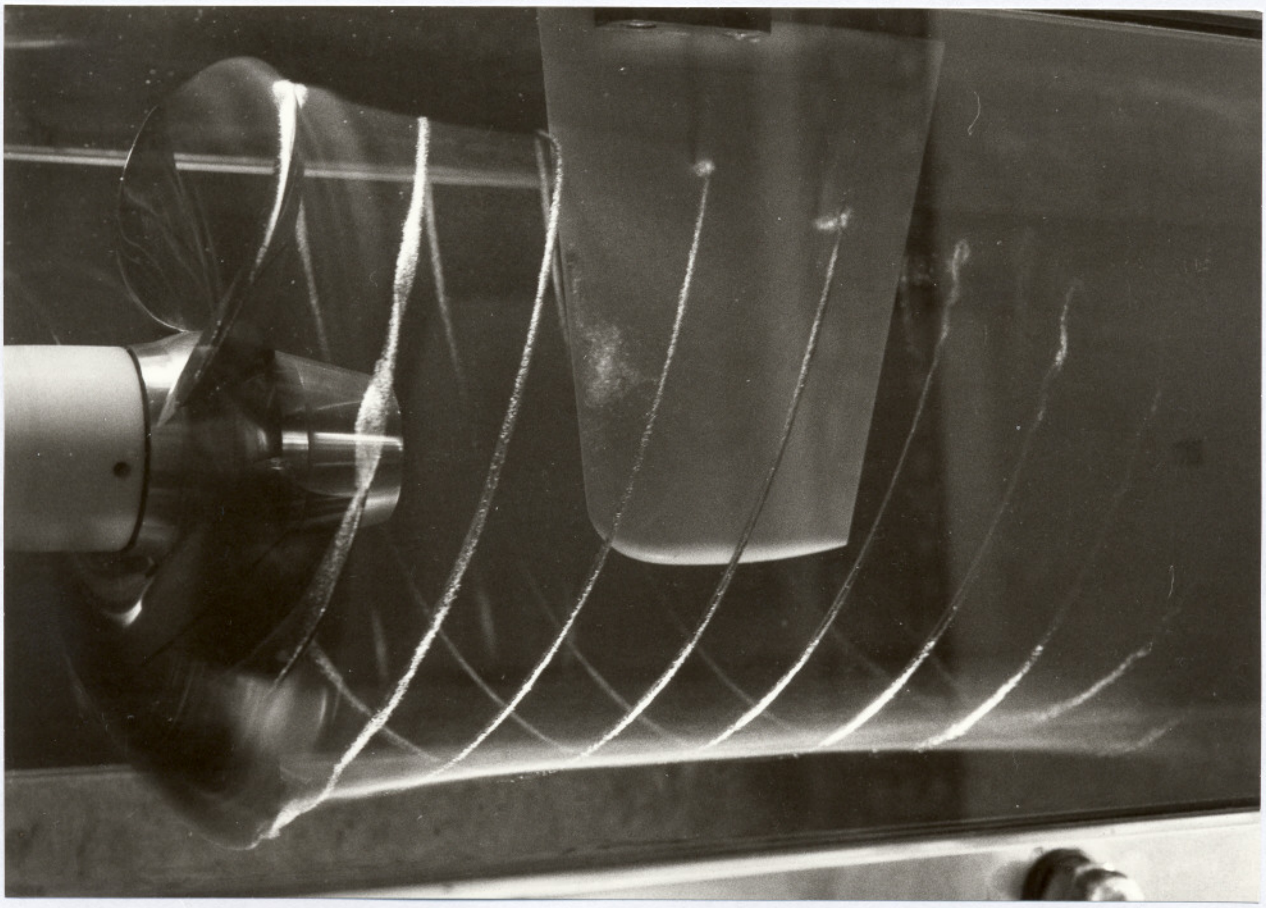
87. Attached vortex pair behind an inclined slender body. A long ogive-cylinder is inclined at 30° to water flowing at 4 cm/s. At this angle of attack a symmetric pair of vortices forms on the lee side of the body. Colored fluid emitted under slight pressure from 0.3-mm holes spirals around the core of the nearer vortex. The Reynolds number is 400 based on the diameter of 1 cm. *Fiechter 1969*



BERLIN

Rudder half in race of a propeller
Starboard view, rudder 12.5° to port

Fig. 7



BERLIN

Rudder half in race of a propeller
Port view, slipstream contraction

Fig.8