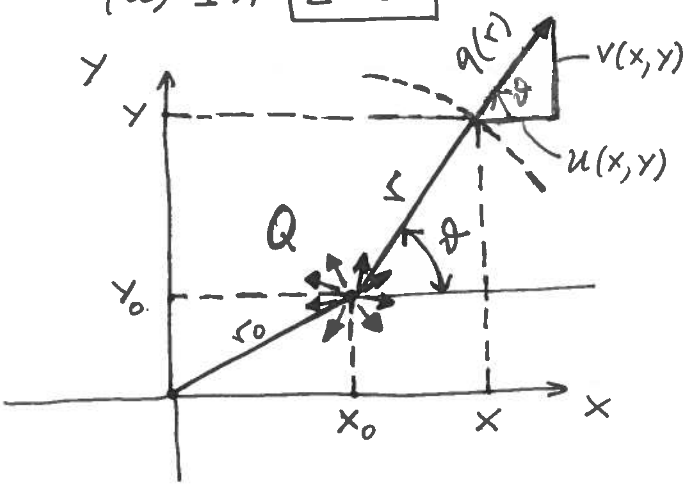
FLOW DUE TO SOURCE (SINK)

(a) In $\boxed{2\text{-D}}$:



Source at point (x_0, y_0) with strength Q induces a radial velocity field $q(r)$

with $r = \sqrt{(x-x_0)^2 + (y-y_0)^2}$

Due to conservation of mass:

$$2\pi r \cdot q(r) = Q \rightsquigarrow$$

$$\rightsquigarrow \boxed{q(r) = \frac{Q}{2\pi r}} \quad (53) \qquad \text{if } Q<0 \longrightarrow \text{SINK}$$

$$u(x,y) = q(r)\cos\vartheta = \frac{Q}{2\pi r}\cdot\frac{x-x_0}{r} = \frac{Q\cdot(x-x_0)}{2\pi\left[(x-x_0)^2+(y-y_0)^2\right]}$$

$$v(x,y) = q(r)\sin\vartheta = \frac{Q}{2\pi r}\,\frac{y-y_0}{r} = \frac{Q\cdot(y-y_0)}{2\pi\left[(x-x_0)^2+(y-y_0)^2\right]}$$

The potential $G(r)$ will be such that:

$$\frac{dG}{dr} = q(r) = \frac{Q}{2\pi r} \longrightarrow G(r) = \int \frac{Q}{2\pi r}\,dr + C = \frac{Q}{2\pi}\ln r$$

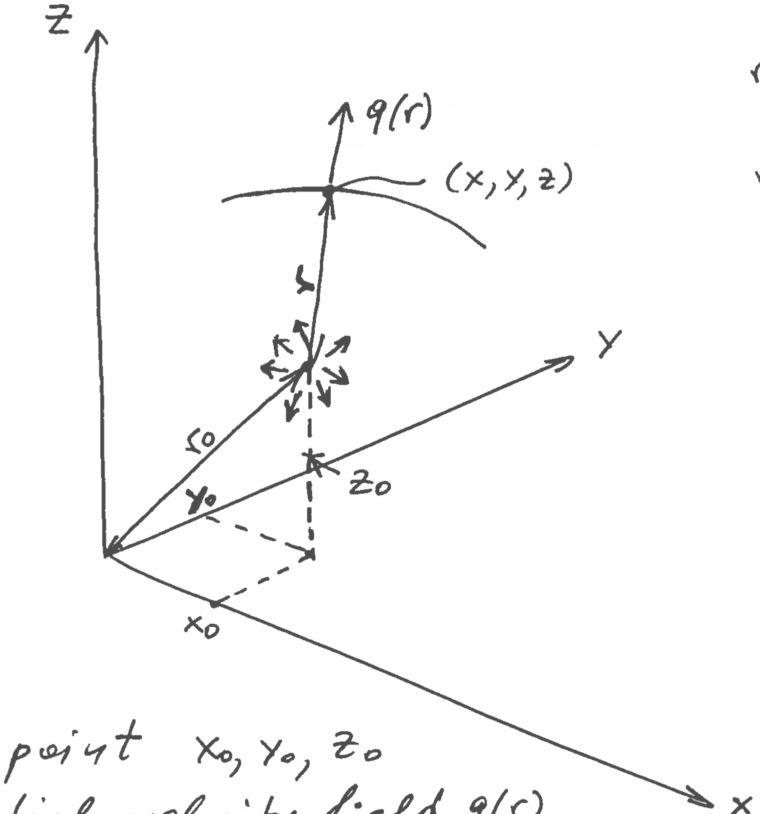
we take ($C=0$)

$$\boxed{G(r) = \frac{Q}{2\pi}\ln r} \quad (54)$$

It can be shown that $\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = \nabla^2 G = 0$ for $r \neq 0$

In fact : $\boxed{\nabla^2 G = Q\,\delta(x-x_0)\,\delta(y-y_0)}$ (55)

(b) Source (sink) in 3-D:



$$r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$$

$$r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

Source Q at point x_0, y_0, z_0 induces radial velocity field $q(r)$

Conservation of mass: $4\pi r^2 q(r) = Q \longrightarrow$

$$\longrightarrow \boxed{q(r) = \frac{Q}{4\pi r^2}} \quad (56)$$

$$(u, v, w) = \frac{Q}{4\pi r^3}\left(x - x_0,\ y - y_0,\ z - z_0\right) \quad (56a)$$

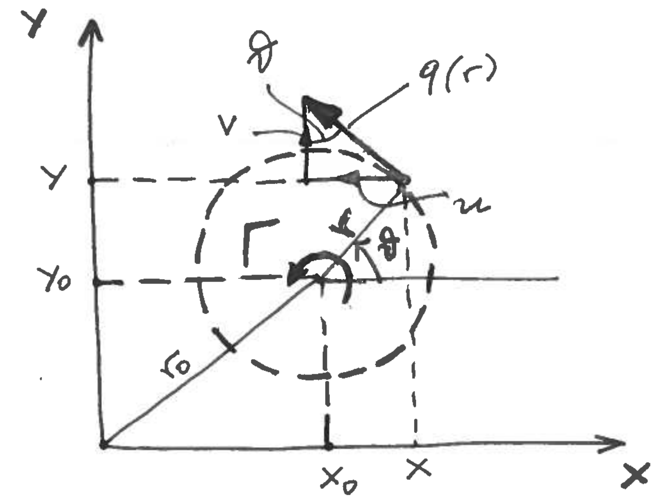
The potential $G(r)$ will be such that:

$$\frac{dG}{dr} = q(r) = \frac{Q}{4\pi r^2} \longrightarrow \boxed{G(r) = -\frac{Q}{4\pi r}} \quad (57)$$

It can be shown that:

$$\boxed{\nabla^2 G = Q \cdot \delta(x - x_0)\,\delta(y - y_0)\,\delta(z - z_0)} \quad (58)$$

FLOW DUE TO A VORTEX (2-D):



Vortex at point (x_0, y_0) with strength Γ induces a circumferential velocity field $q(r)$

$$r = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

Due to irrotational flow (everywhere except where the vortex is located) we have:

$$2\pi r\, q(r) = \Gamma \;\rightsquigarrow$$

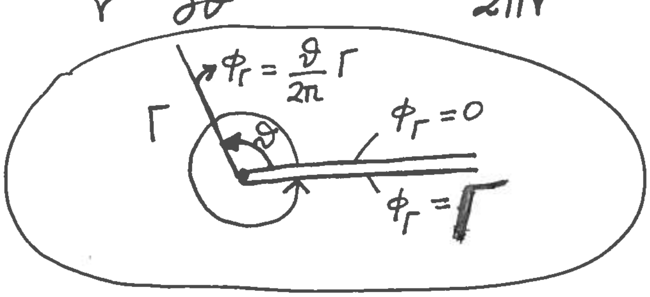
$$\rightsquigarrow \quad \boxed{q(r) = \frac{\Gamma}{2\pi r}} \qquad (55a)$$

$$u(x,y) = -q(r)\sin\vartheta = \frac{-\Gamma}{2\pi r}\cdot\frac{y-y_0}{r} = \frac{-\Gamma\,(y-y_0)}{2\pi\left[(x-x_0)^2+(y-y_0)^2\right]}$$

$$v(x,y) = q(r)\cos\vartheta = \frac{\Gamma}{2\pi r}\cdot\frac{x-x_0}{r} = \frac{\Gamma\,(x-x_0)}{2\pi\left[(x-x_0)^2+(y-y_0)^2\right]}$$

The potential $\phi_\Gamma(\vartheta)$ will be such that:

$$\frac{1}{r}\frac{\partial\phi_\Gamma}{\partial\vartheta} = q(r) = \frac{\Gamma}{2\pi r} \longrightarrow \frac{\partial\phi_\Gamma}{\partial\vartheta} = \frac{\Gamma}{2\pi} \longrightarrow \boxed{\phi_\Gamma = \Gamma\frac{\vartheta}{2\pi}} \qquad (55b)$$



It can be shown that $\boxed{\nabla^2\phi_\Gamma = 0}$ (55c)