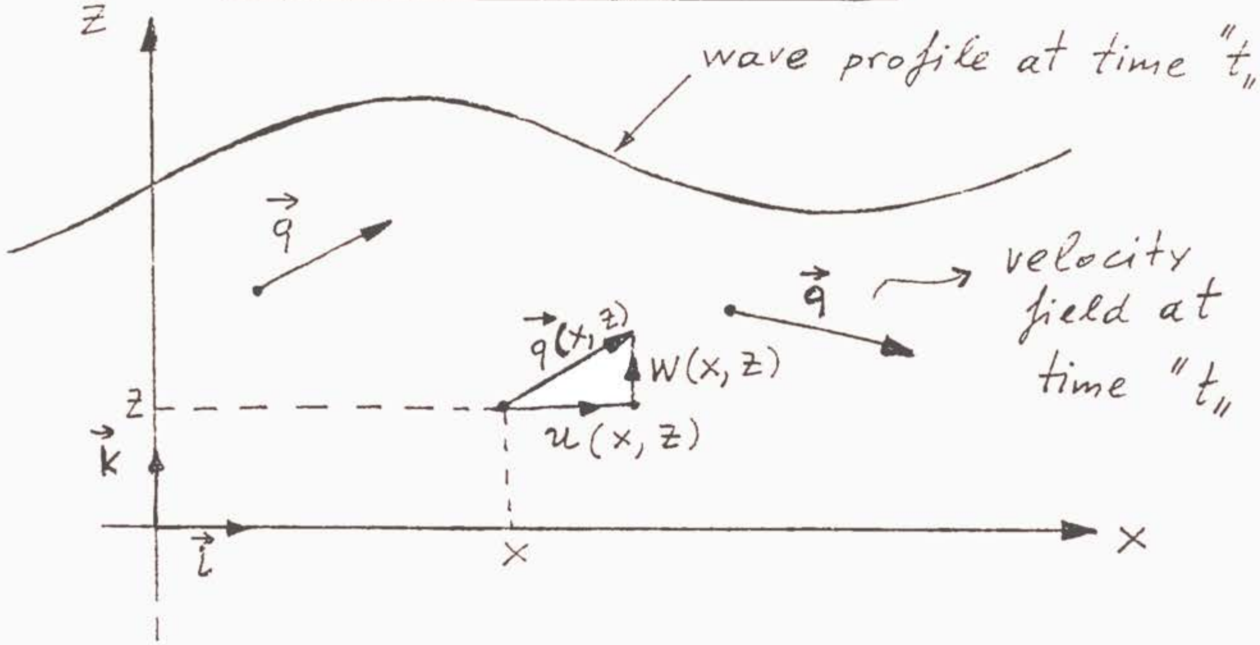
FLUID MECHANICS

INCOMPRESSIBLE FLUIDS



$\vec{q}$ = particle velocity

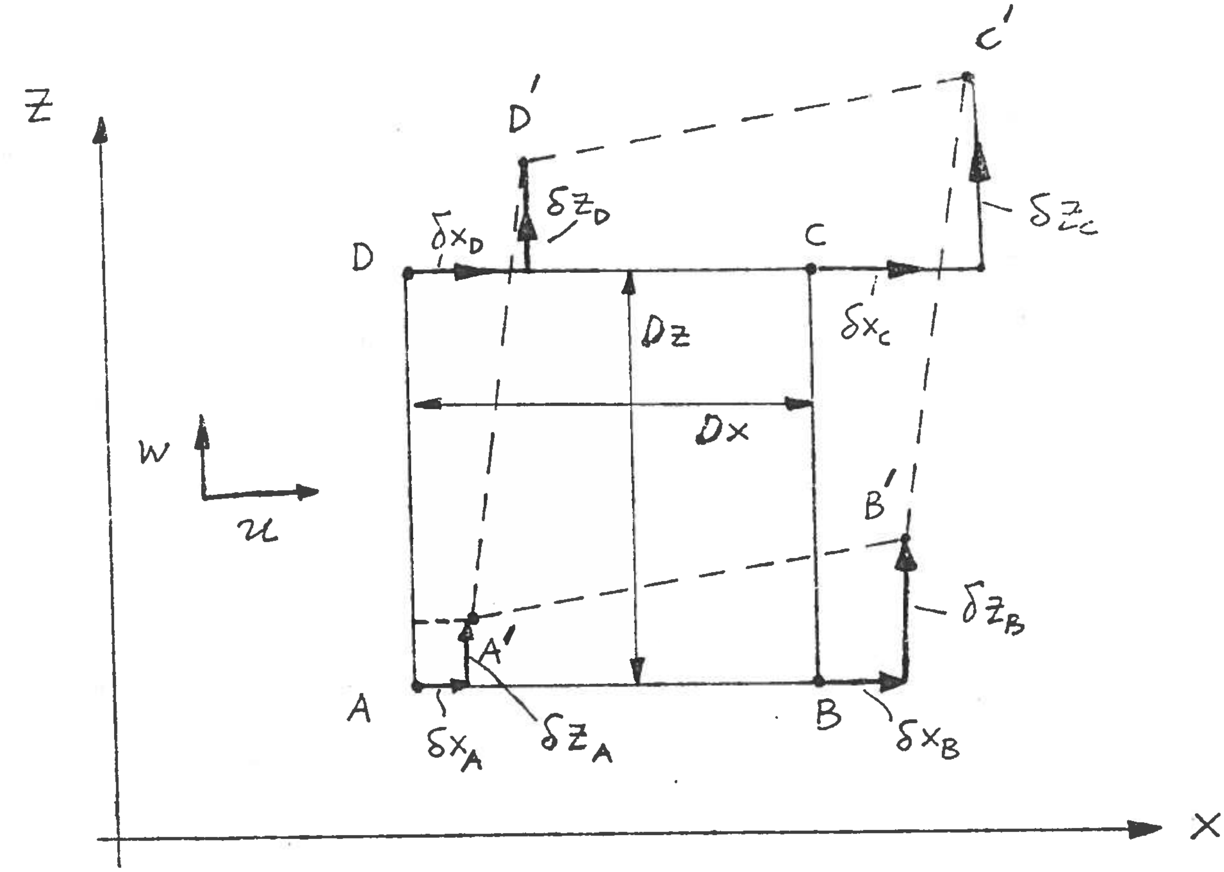
u = horizontal velocity component

w = vertical " "

$$\vec{q} = u \cdot \vec{i} + w \cdot \vec{k}$$

$\vec{i}, \vec{k}$: unit vectors along x, z axes, respectively

Q: Can u and w be arbitrary functions of (x, z)? What laws should they obey?



Consider 2-D flow with "unit" depth.
Fluid "volume" ABCD at time "t" moves to new location A'B'C'D' at time $t' = t + \delta t$

For example point A moves to A':

$$x_{A'} = x_A + \delta x_A = x_A + u_A \, \delta t$$

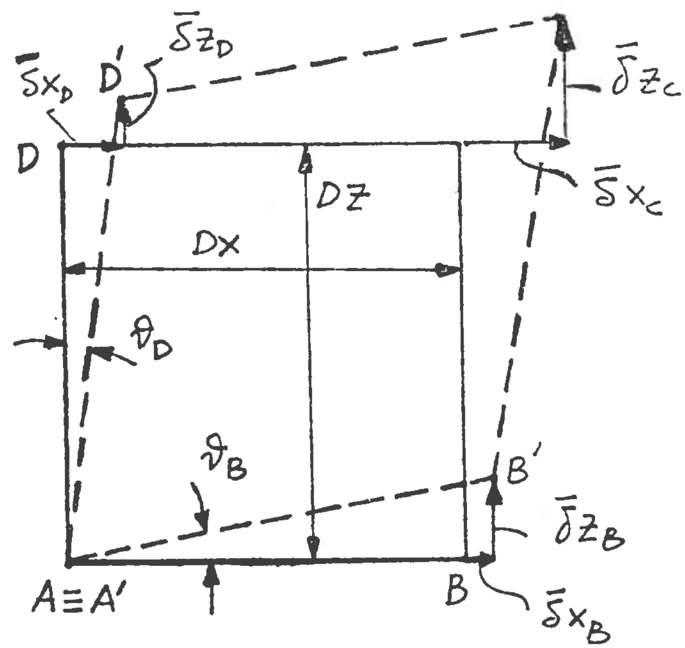
$$z_{A'} = z_A + \delta z_A = z_A + w_A \, \delta t$$

Similarly:

$$x_{B'} = x_B + \delta x_B = x_B + u_B \, \delta t$$

$$z_{B'} = z_B + \delta z_B = z_B + w_B \, \delta t$$

... etc.



Then the relative motions are:

$$\begin{cases} \bar{\delta} x_B = \delta x_B - \delta x_A = u_B \delta t - u_A \delta t = (u_B - u_A)\delta t \\ \bar{\delta} z_B = \delta z_B - \delta z_A = w_B \delta t - w_A \delta t = (w_B - w_A)\delta t \end{cases}$$

$$\begin{cases} \bar{\delta} x_C = \delta x_C - \delta x_A = u_C \delta t - u_A \delta t = (u_C - u_A)\delta t \\ \bar{\delta} z_C = \delta z_C - \delta z_A = w_C \delta t - w_A \delta t = (w_C - w_A)\delta t \end{cases}$$

$$\begin{cases} \bar{\delta} x_D = \delta x_D - \delta x_A = u_D \delta t - u_A \delta t = (u_D - u_A)\delta t \\ \bar{\delta} z_D = \delta z_D - \delta z_A = w_D \delta t - w_A \delta t = (w_D - w_A)\delta t \end{cases}$$

Now remember definitions of partial derivatives!
Then the above relative motions may be approximated as follows:

$$\bar{\delta} x_B \approx \frac{\partial u}{\partial x} Dx \, \delta t$$

since $u_B \approx u_A + \frac{\partial u}{\partial x}(x_B - x_A) = u_A + \frac{\partial u}{\partial x} Dx$

Similarly:

$$\bar{\delta} z_B \approx \frac{\partial w}{\partial x} Dx \, \delta t$$

$$\bar{\delta} x_c \approx \left[\frac{\partial u}{\partial x} Dx + \frac{\partial u}{\partial z} Dz \right] \delta t$$

since $u_c \approx u_A + \frac{\partial u}{\partial x}(x_c - x_A) + \frac{\partial u}{\partial z}(z_c - z_A) =$

$$= u_A + \frac{\partial u}{\partial x} Dx + \frac{\partial u}{\partial z} Dz$$

$$\bar{\delta} z_c \approx \left[\frac{\partial w}{\partial x} Dx + \frac{\partial w}{\partial z} Dz \right] \delta t$$

$$\bar{\delta} x_D \approx \frac{\partial u}{\partial z} Dz \, \delta t$$

$$\bar{\delta} z_D \approx \frac{\partial w}{\partial z} Dz \, \delta t$$

From the above it can be easily seen that:

$$\bar{\delta} x_c \approx \bar{\delta} x_B + \bar{\delta} x_D$$

$$\bar{\delta} z_c \approx \bar{\delta} z_B + \bar{\delta} z_D$$

Physical restrictions on relative motions:

(a) Conservation of mass (volume) of the rectangle ABCD

For $\bar{\delta} z_B, \bar{\delta} x_B, \bar{\delta} x_D, \bar{\delta} z_D \ll Dx, Dz$ (i.e. for δt "small") the area of $A'B'C'D'$ may be approximated from:

$$E' \approx A'D' \cdot A'B'$$

Also the lengths of segments $A'D'$ and $A'B'$ may be approximated as:

$$A'B' \approx Dx + \bar{\delta} x_B$$

$$A'D' \approx Dz + \bar{\delta} z_D$$

Thus: $E' \approx (Dx + \bar{\delta} x_B)(Dz + \bar{\delta} z_D) =$

$$= Dx \cdot Dz + Dz\,\bar{\delta} x_B + Dx\,\bar{\delta} z_D + \underbrace{\bar{\delta} x_B \bar{\delta} z_D}_{\text{"small"}} \approx$$

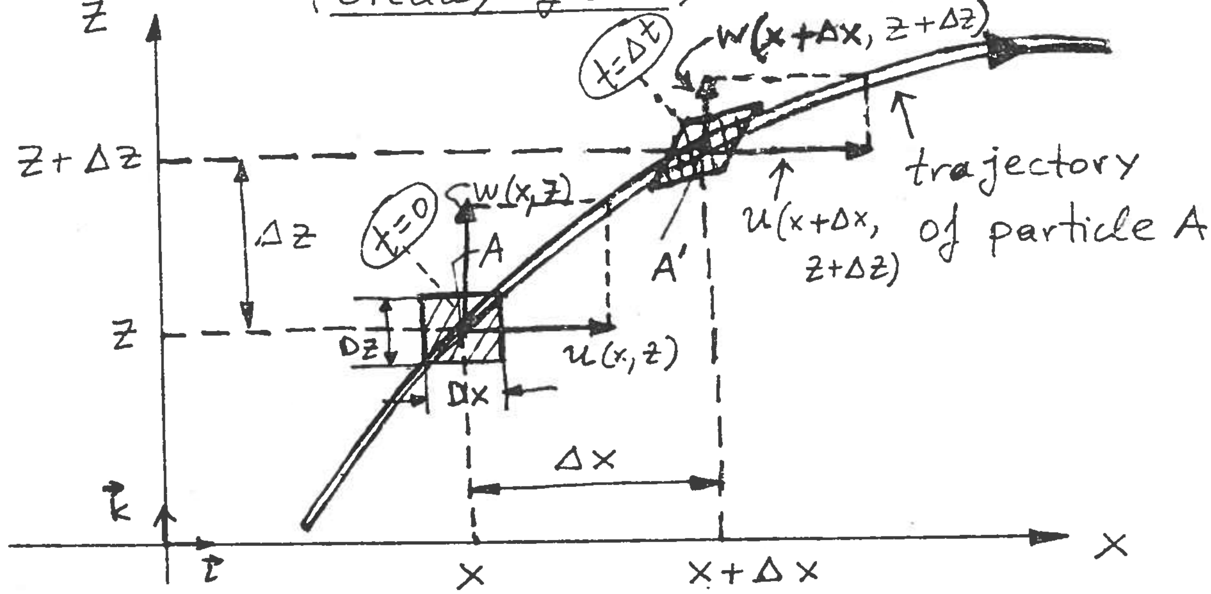
$$\approx Dx \cdot Dz + Dz \frac{\partial u}{\partial x} Dx\,\delta t + Dx \frac{\partial w}{\partial z} Dz\,\delta t$$

Remember that $E = Dx \cdot Dz$ is the area of ABCD
So in order to have $E' = E$ we must have:

Equ. (26) $\boxed{\dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0}$ Conservation of mass equ also called Continuity equation

ACCELERATION OF A FLUID PARTICLE: (2-D)

(Steady flow)



Particle A (of mass $Dm = \rho\, Dx\, Dz$, ρ: density) is at location (x, z) at time $t=0$.
At time $t=\Delta t$ moves to new location A' $(x+\Delta x, z+\Delta z)$.

$$\text{Acceleration } \vec{a} = a_x \cdot \vec{i} + a_z \cdot \vec{k} = \frac{\Delta \vec{q}}{\Delta t} =$$

$$= \frac{\Delta(u.\vec{i} + w.\vec{j})}{\Delta t} = \frac{\Delta u}{\Delta t} \cdot \vec{i} + \frac{\Delta W}{\Delta t} \cdot \vec{k}$$

$$\text{so} \quad a_x = \frac{\Delta u}{\Delta t}, \quad a_z = \frac{\Delta W}{\Delta t}$$

$$a_x = \frac{\Delta u}{\Delta t} = \frac{u(x+\Delta x, z+\Delta z) - u(x, z)}{\Delta t} =$$

$$= \frac{\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial z}\Delta z}{\Delta t}$$

But: $\boxed{\begin{array}{l} \Delta x = u \cdot \Delta t \\ \Delta z = w \cdot \Delta t \end{array}}$

So: $a_x = u \cdot \frac{\partial u}{\partial x} + w \cdot \frac{\partial u}{\partial z}$

Similarly $a_z = u \cdot \frac{\partial w}{\partial x} + w \cdot \frac{\partial w}{\partial z}$

In the event u, w are also functions of time (unsteady flow) then:

$$a_x = \frac{\partial u}{\partial t} + u \cdot \frac{\partial u}{\partial x} + w \cdot \frac{\partial u}{\partial z}$$

equ.(27) $$a_z = \underbrace{\frac{\partial w}{\partial t}}_{\text{unsteady terms}} + \underbrace{u \cdot \frac{\partial w}{\partial x} + w \cdot \frac{\partial w}{\partial z}}_{\text{convective terms}}$$

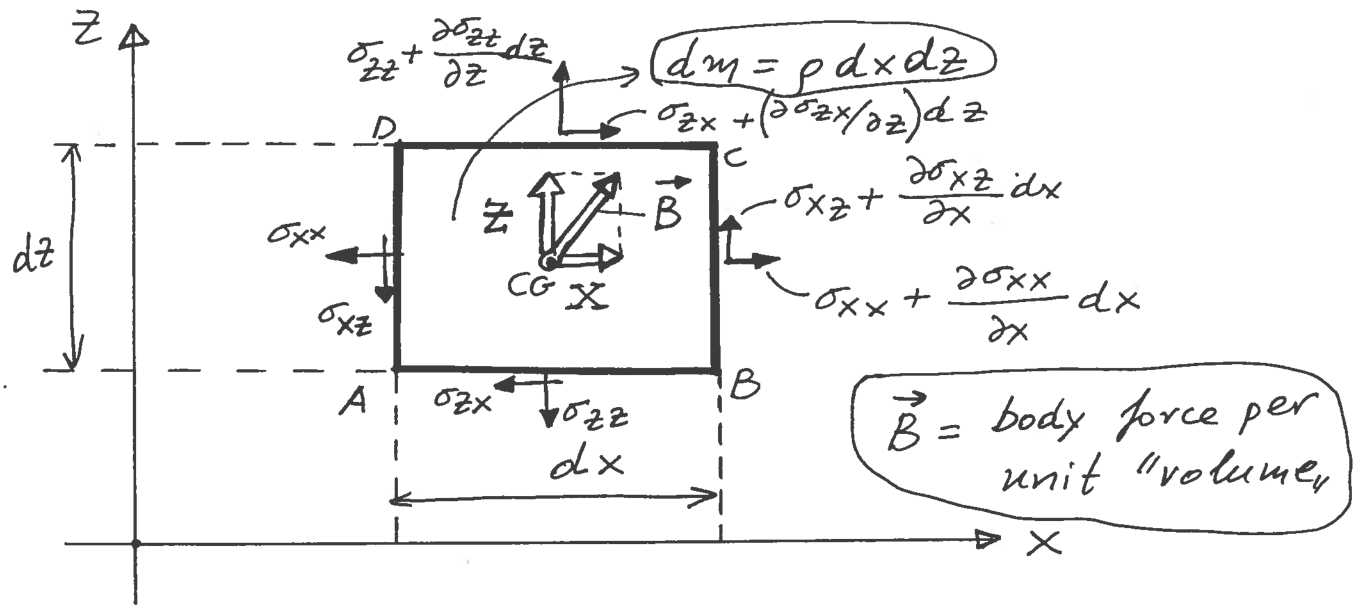
$\boxed{\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}}$

D/Dt operator in 3-D

$\boxed{\vec{a} = \frac{D\vec{q}}{Dt}}$

Definition of substantial derivative (or total or material)

FORCES ON FLUID PARTICLE (2-D)



Mass of fluid particle (ABCD) $\underline{dm = \rho\, dx\, dz}$

Force along x: $dF_x = \left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{zx}}{\partial z} + X \right] dx\, dz$

" z: $dF_z = \left[\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} + Z \right] dx\, dz$

- In the case of solids [p.10] we required equilibrium of forces $\Longrightarrow dF_x = dF_z = 0$
- In the case of fluids though the particle has acceleration. Thus we need to apply Newton's Law:

NEWTON'S LAW APPLIED ON FLUID PARTICLE:

$$dF_x = dm \cdot a_x$$
$$dF_z = dm \cdot a_z$$

Thus, replacing $dF_{x,z}$, $a_{x,z}$ from the expressions developed earlier, we get:

$$\rho \frac{Du}{Dt} \equiv \rho \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial w}{\partial z} \right] = \left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{zx}}{\partial z} + X \right] \quad 28(a)$$

$$\rho \frac{Dw}{Dt} \equiv \rho \left[\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right] = \left[\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} + Z \right] \quad 28(b)$$

In the event gravity is towards $-z$

$$X = 0, \quad Z = -\rho g$$

- Conservation of angular momentum leads to: $\boxed{\sigma_{xz} = \sigma_{zx}}$

CONTINUITY AND NEWTON'S LAW IN 3-D

(a) Continuity: (incompressible fluid)

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0} \quad (29)$$

or $\text{div.}\,\vec{q}$ or $\boxed{\vec{\nabla}\cdot\vec{q} = 0}$

(b) Newton's Law:

$$\rho\frac{Du}{Dt} = \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] = \left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + X\right] \quad 30(a)$$

$$\rho\frac{Dv}{Dt} = \rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right] = \left[\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + Y\right] \quad 30(b)$$

$$\rho\frac{Dw}{Dt} = \rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right] = \left[\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + Z\right] \quad 30(c)$$

$$\sigma_{xz} = \sigma_{zx}\,, \quad \sigma_{xy} = \sigma_{yx}\,, \quad \sigma_{yz} = \sigma_{zy}$$

In a notation where $(x_1, x_2, x_3) \to (x, y, z)$; $(u_1, u_2, u_3) \to (u, v, w)$
and $(X_1, X_2, X_3) \to (X, Y, Z)$

$$\boxed{\rho\frac{Du_i}{Dt} = \frac{\partial \sigma_{ji}}{\partial x_j} + X_i} \quad 30(d)\ (i = 1, 2, 3)$$

$$\frac{\partial \sigma_{ji}}{\partial x_j} \equiv \frac{\partial \sigma_{1i}}{\partial x_1} + \frac{\partial \sigma_{2i}}{\partial x_2} + \frac{\partial \sigma_{3i}}{\partial x_3} \quad \text{(Einstein notation)}$$

CONSTITUTIVE RELATIONS - NEWTONIAN FLUIDS

$$\boxed{\sigma_{ij} = -p\delta_{ij} + 2\mu\dot{\varepsilon}_{ij}} \qquad (31)$$

$(1 \to x,\ 2 \to y,\ 3 \to z)$

Compare (31) to equations (11) for solids.

μ = (coefficient of) viscosity

(also called dynamic viscosity)

viscosity corresponds to Lamé's constant μ (with different units though)

$\dot{\varepsilon}_{ij}$ = rate of strain; $\dot{\varepsilon}_{ij} = \frac{1}{2}\left[\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i}\right]$

(u_i, u_j = displacements
$\dot{u}_i, \dot{u}_j$ = velocities)

Note: $\boxed{\dot{\Theta} = \dot{\varepsilon}_{ii} = \frac{\partial \dot{u}_i}{\partial x_i} = \vec{\nabla}\cdot\vec{q} = 0}$

In (2-D) eq. (31) becomes:

$$\boxed{\begin{aligned} \sigma_{xx} &= -p + 2\mu\frac{\partial u}{\partial x} \\ \sigma_{zz} &= -p + 2\mu\frac{\partial w}{\partial z} \\ \sigma_{xz} = \sigma_{zx} &= \mu\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] \end{aligned}} \qquad (32)$$

The RHS of 28(a) becomes: $-\frac{\partial p}{\partial x} + 2\mu\left[\frac{\partial^2 u}{\partial x^2}\right] + \mu\left[\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z}\right]$
$(X = 0)$

However from continuity $\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} = 0 \to \frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} \to$

$\to \frac{\partial^2 w}{\partial z \partial x} = -\frac{\partial^2 u}{\partial x^2} \Rightarrow$ RHS of 28(a): $-\frac{\partial p}{\partial x} + \mu\nabla^2 u$

NAVIER-STOKES EQUATIONS:

(3-D): Following the analysis of the previous page we get: (for incompressible fluids)

$$\boxed{\begin{aligned} \rho \frac{Du}{Dt} &= -\frac{\partial p}{\partial x} + \mu \nabla^2 u \\ \rho \frac{Dv}{Dt} &= -\frac{\partial p}{\partial y} + \mu \nabla^2 v \\ \rho \frac{Dw}{Dt} &= -\frac{\partial p}{\partial z} + \mu \nabla^2 w \end{aligned}} \quad (33) \quad + \vec{B}$$

(body force per unit volume)

($\vec{B} = \rho \vec{g}$ for gravity)

In more compact form:

$$\boxed{\rho \frac{D\vec{q}}{Dt} = -\vec{\nabla} p + \mu \nabla^2 \vec{q}} \quad (34) \quad + \vec{B}$$

or

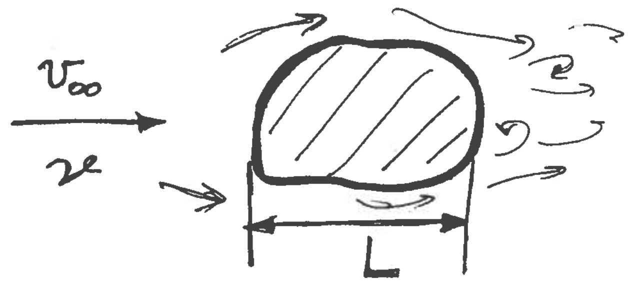
$$\boxed{\frac{D\vec{q}}{Dt} = -\frac{\vec{\nabla} p}{\rho} + \nu \nabla^2 \vec{q}} \quad (35) \quad + \frac{\vec{B}}{\rho}$$

($= \vec{g}$ for gravity)

With $\boxed{\nu = \frac{\mu}{\rho}}$ <u>kinematic</u> <u>viscosity</u>

$\boxed{\text{Note:}}$ The body force needs to be added to (33), (34), (35) if it is not negligible.

THE REYNOLDS NUMBER:



$$\boxed{Re = \frac{v_\infty . L}{\nu}}$$

with gravity included

N-S Equations are:

$$\frac{D\vec{q}}{Dt} = -\frac{\vec{\nabla} p}{\rho} + \nu \nabla^2 \vec{q} \qquad (35) \quad + \vec{g}$$

Define: $\vec{q}\,' = \dfrac{\vec{q}}{v_\infty}$, $(x', y', z') = \dfrac{1}{L}(x, y, z)$, $t' = \dfrac{t}{(L/v_\infty)}$

$$p' = \frac{p}{\rho v_\infty^2}$$

Then (35) becomes: $\left(\vec{\nabla} = \frac{1}{L}\vec{\nabla}'\right)$

$$\frac{v_\infty^2}{L}\frac{D\vec{q}\,'}{Dt'} = -\frac{\vec{\nabla}' p'}{\rho}\,\rho v_\infty^2/L + \nu . \frac{v_\infty}{L^2}\nabla'^2 \vec{q}\,' \quad + \vec{g}$$

$$\sim \boxed{\frac{D\vec{q}\,'}{Dt'} = -\vec{\nabla}' p' + \frac{1}{Re}\nabla'^2 \vec{q}\,'} \qquad (36) \quad + \frac{gL}{v_\infty^2}\frac{\vec{g}}{g}$$

$$\frac{gL}{v_\infty^2} \rightarrow \frac{1}{F_r^2}$$

$$\boxed{Re = \text{Reynolds \#} = \frac{v_\infty . L}{\nu}} \qquad (37)$$

Froude #

$$F_r = \frac{v_\infty}{\sqrt{gL}}$$

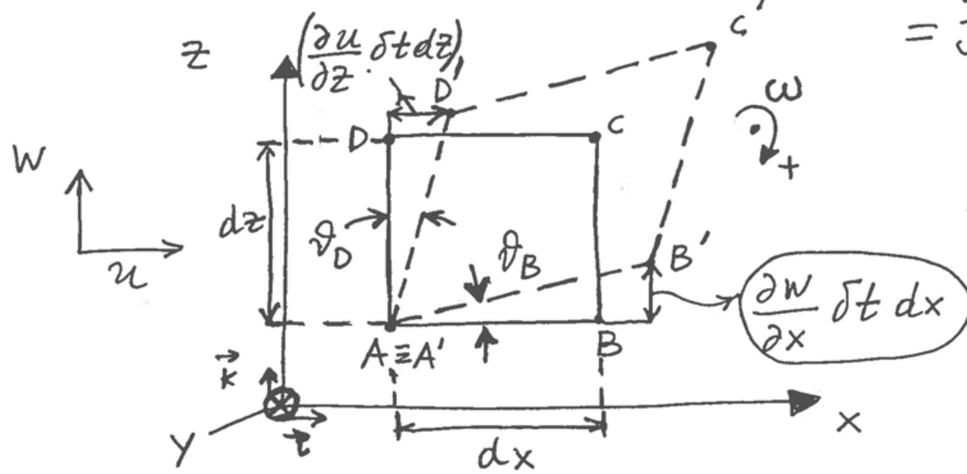
VORTICITY VECTOR, $\vec{\omega}$

Definition: $$\boxed{\vec{\omega} = \vec{\nabla} \times \vec{q}} \qquad (38)$$

$$\vec{\omega} = \vec{\nabla} \times \vec{q} \underset{\text{DEF}}{\equiv} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

In (2-D) (see also p. 29) $\quad \vec{\omega} = -\vec{j}\left[\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right]$

$(v = 0,\ \partial/\partial y = 0)$



$$= \vec{j}\left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right]$$

$$\boxed{\omega = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}} \quad \text{(2-D)} \qquad (39)$$

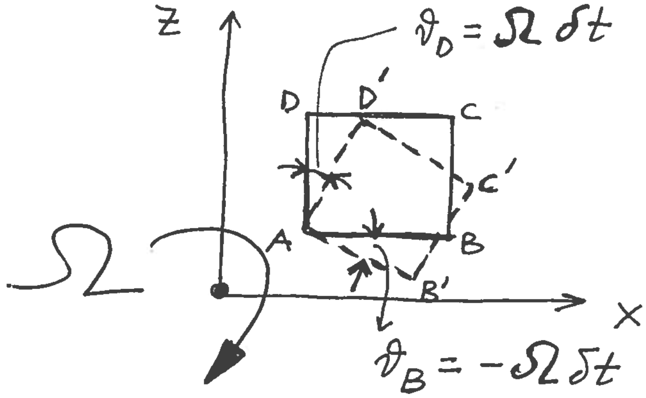
Note that: $$\vartheta_B = \frac{\frac{\partial w}{\partial x}\delta t\, dx}{dx} = \frac{\partial w}{\partial x}\delta t$$

$$\vartheta_D = \frac{\frac{\partial u}{\partial z}\delta t\, dz}{dz} = \frac{\partial u}{\partial z}\delta t$$

$$\omega = \left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right] = \frac{(\vartheta_D - \vartheta_B)}{\delta t}$$

SPECIAL CASES:

(a) Pure rotation:

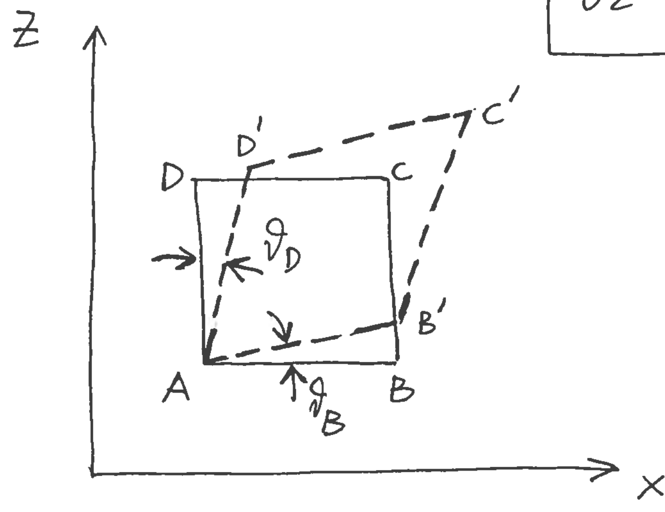


$$\underline{\underline{\omega}} = \frac{\vartheta_D - \vartheta_B}{\delta t} = \frac{\Omega\delta t + \Omega\delta t}{\delta t} = \underline{\underline{2\Omega}}$$

$$\boxed{\omega = 2\Omega}$$

(b) Irrotational flow:

$$\omega = 0 \rightsquigarrow \vartheta_D = \vartheta_B \qquad \boxed{\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}} \qquad (40)$$



VORTICITY EQUATION:

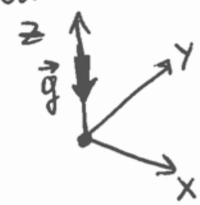
Equations of fluid motion: (incompressible)

Continuity: $\vec{\nabla}\cdot\vec{q} = 0 \quad \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\right)$

$$\vec{\nabla}\cdot\vec{q} \equiv \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(\vec{i}u + \vec{j}v + \vec{k}w\right)$$

N-S equations: $\frac{D\vec{q}}{Dt} = -\frac{\vec{\nabla}p}{\rho} + \nu\nabla^2\vec{q} \quad \boxed{+\vec{g} = -\vec{\nabla}(gz)}$

with gravity included



$$\frac{D\vec{q}}{Dt} = \frac{\partial\vec{q}}{\partial t} + \underbrace{(\vec{q}\cdot\vec{\nabla})}\vec{q} \qquad (40a)$$

$$\left(\vec{i}u + \vec{j}v + \vec{k}w\right)\cdot\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)$$

$$\equiv$$

$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)$$

Note that $\vec{\nabla}\cdot\vec{q} \neq \vec{q}\cdot\vec{\nabla}$

$$\vec{\nabla}p = \left[\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right]p = \vec{i}\frac{\partial p}{\partial x} + \vec{j}\frac{\partial p}{\partial y} + \vec{k}\frac{\partial p}{\partial z}$$

$$\nabla^2\vec{q} = (\vec{\nabla}\cdot\vec{\nabla})\vec{q} = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\vec{q}$$

From vector analysis: ($\vec{u}, \vec{v}$ vectors)

$$\boxed{\vec{\nabla}(\vec{u}\cdot\vec{v}) = (\vec{u}\cdot\vec{\nabla})\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{u} + \vec{u}\times(\vec{\nabla}\times\vec{v}) + \vec{v}\times(\vec{\nabla}\times\vec{u})} \quad (41)$$

Applying (41) with $\vec{u} = \vec{v} = \vec{q}$ and using (38) we get:

$$\boxed{(\vec{q}\cdot\vec{\nabla})\vec{q} = \vec{\omega}\times\vec{q} + \vec{\nabla}\left(\frac{q^2}{2}\right)} \quad (41a)$$

Thus (40a) becomes:

$$\boxed{\frac{D\vec{q}}{Dt} = \frac{\partial\vec{q}}{\partial t} + \vec{\omega}\times\vec{q} + \vec{\nabla}\left(\frac{q^2}{2}\right)} \quad (41b)$$

So, the N-S equations become:

$$\frac{\partial\vec{q}}{\partial t} + \vec{\omega}\times\vec{q} + \vec{\nabla}\left(\frac{q^2}{2}\right) = -\vec{\nabla}\left(\frac{p}{\rho}\right) + \nu\nabla^2\vec{q} \quad (41c)$$

Taking the $\underline{\underline{\vec{\nabla}\times}}$ of both sides of (41c) we get:

$$\frac{\partial\vec{\omega}}{\partial t} + \vec{\nabla}\times(\vec{\omega}\times\vec{q}) = \nu\nabla^2\vec{\omega} \quad (41d)$$

where we have used the identity:

$$\vec{\nabla}\times\left[\vec{\nabla}(f)\right] = 0 \qquad (f: \text{scalar})$$

Using the identity:

$$\boxed{\vec{\nabla}\times(\vec{u}\times\vec{v}) = (\vec{v}\cdot\vec{\nabla})\vec{u} - (\vec{u}\cdot\vec{\nabla})\vec{v} + \vec{u}(\vec{\nabla}\cdot\vec{v}) - \vec{v}(\vec{\nabla}\cdot\vec{u})} \quad (42)$$

with $\vec{u}=\vec{\omega}$ and $\vec{v}=\vec{q}$

and the identity: $\underline{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{u})=0}$ (i.e. $\underline{\underline{\vec{\nabla}\cdot\vec{\omega}=0}}$)

and the continuity condition $\vec{\nabla}\cdot\vec{q}=0$

we get: $\underline{\vec{\nabla}\times(\vec{\omega}\times\vec{q}) = (\vec{q}\cdot\vec{\nabla})\vec{\omega} - (\vec{\omega}\cdot\vec{\nabla})\vec{q}}$ (42a)

Substituting (42a) in (41d) we get:

$$\frac{\partial\vec{\omega}}{\partial t} + (\vec{q}\cdot\vec{\nabla})\vec{\omega} = (\vec{\omega}\cdot\vec{\nabla})\vec{q} + \nu\nabla^2\vec{\omega}$$

or

vortex stretching (only in 3-D)

vortex diffusion

$$\boxed{\frac{D\vec{\omega}}{Dt} = (\vec{\omega}\cdot\vec{\nabla})\vec{q} + \nu\nabla^2\vec{\omega}} \quad (43)$$

Vorticity equation

(for incompressible flow)

INVISCID – IRROTATIONAL FLOWS:

For $\underline{\nu = 0}$ (or to be more accurate for $\nu \nabla^2 \vec{q} = 0$ or $\nu \nabla^2 \vec{\omega} = 0$)

equ. (43) becomes:

$$\underline{\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \vec{\nabla})\, \vec{q}} \quad (44)$$

It can be shown that if $\vec{\omega} = 0$ $t \leq 0$ then based on (44): $\boxed{\vec{\omega} = 0}$ at all times. In other words if an inviscid fluid is irrotational at a given moment, will continue to be irrotational.

In 2-D equation (43) becomes:

$$\underline{\frac{D\omega}{Dt} = \nu \nabla^2 \omega} \quad (45) \quad \left(\omega = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$\longrightarrow$ Diffusion equation for ω

For inviscid 2-D flows: $\dfrac{D\omega}{Dt} = 0$

$\rightsquigarrow$ vorticity is convected with the flow without changing its magnitude.

Q: How is vorticity generated?

A: Close to the boundaries where (due to no-slip condition) $\nu \nabla^2 \vec{q}$ or $\nu \nabla^2 \vec{\omega}$ is not negligible

In 2-D, irrotational flows must satisfy:

(a) $\omega = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0$ (irrotationality condition)

and

(b) $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$ (continuity condition)

Expressing $u = \frac{\partial \phi}{\partial x}$, $w = \frac{\partial \phi}{\partial z}$ then

(a) $\rightarrow \frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x} = 0 \longrightarrow$ $\omega = 0$ automatically

Then (b) must also be satisfied

$$\sim \quad \boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 = \nabla^2 \phi} \qquad (46)$$

In 3-D with $\vec{\omega} = 0 \Rightarrow \vec{\nabla} \times \vec{q} = 0 \longrightarrow$

$\rightarrow$ There exists a scalar function ϕ such that

$$\boxed{\vec{q} = \vec{\nabla} \phi} \qquad (47)$$

In order for $\vec{\nabla} \cdot \vec{q} = 0$ (continuity equ.) $\Longrightarrow$ Laplace's equation $\boxed{\nabla^2 \phi = 0}$ (48)

$$\Updownarrow$$

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0}$$

$\boxed{\phi : \text{is the velocity potential}}$

(47) $\Leftrightarrow u = \frac{\partial \phi}{\partial x}$, $v = \frac{\partial \phi}{\partial y}$, $w = \frac{\partial \phi}{\partial z}$

BERNOULLI'S EQUATION:

For $\vec{\omega}=0$, equ. (41b) gives:

$$\frac{D\vec{q}}{Dt} = \frac{\partial \vec{q}}{\partial t} + \vec{\nabla}\left(\frac{q^2}{2}\right)$$

The Navier-Stokes (N-S) equations then give: (35)

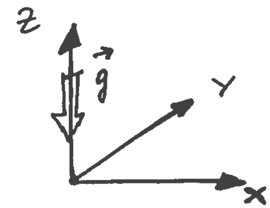
$$\frac{\partial \vec{q}}{\partial t} + \vec{\nabla}\left(\frac{q^2}{2}\right) = -\vec{\nabla}\left(\frac{p}{\rho}\right)$$

IMPORTANT THEOREM

For incompressible flow

$\Rightarrow \nabla^2 \vec{q} = -\vec{\nabla}\times\vec{\omega}$

Using (47) we finally get:



$$\vec{\nabla}\left(\frac{\partial \phi}{\partial t} + \frac{q^2}{2} + \frac{p}{\rho}\right) = 0$$

In the event there is gravity present then $\vec{B} = \rho(0,0,-g) = -\rho\vec{\nabla}(gz)$ must be added to the RHS of N-S equations.

Then: $$\vec{\nabla}\left(\frac{\partial \phi}{\partial t} + \frac{q^2}{2} + \frac{p}{\rho} + gz\right) = 0$$

or $$\boxed{\rho\frac{\partial \phi}{\partial t} + \rho\frac{q^2}{2} + p + \rho g z = \text{constant}} \quad (49)$$

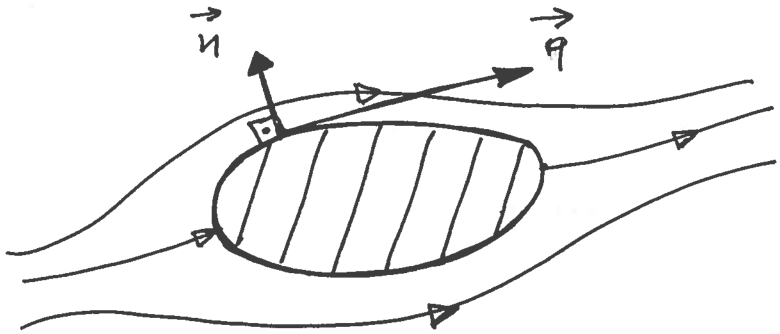
Generalized Bernoulli's equation for unsteady/inviscid/irrotational (incompressible) flows

(if steady flow $\Rightarrow \frac{\partial}{\partial t} = 0 \Rightarrow$ traditional Bernoulli equ.

BOUNDARY CONDITIONS:

(a) Kinematic Boundary Condition:

(For inviscid flow)

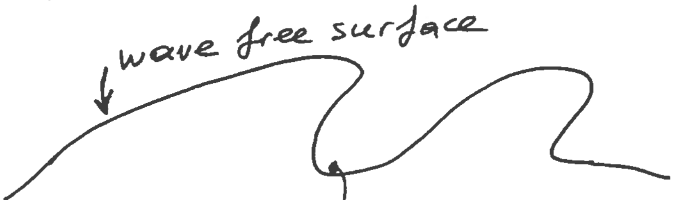


Velocity vector is tangent to solid boundary: (*)

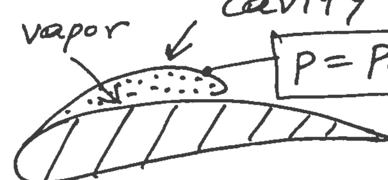
$$\boxed{\vec{q}\cdot\vec{n}=0 \quad \text{or} \quad \frac{\partial\phi}{\partial n}=\vec{\nabla}\phi\cdot\vec{n}=\vec{q}\cdot\vec{n}=0} \qquad (50)$$

(b) Dynamic boundary Condition:

(it involves pressure)



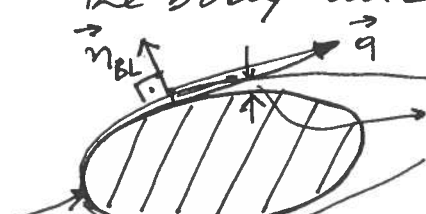
$\boxed{p=P_{atm}}$ at any point and time on the free surface



$\boxed{p=P_{vapor}}$ at any point and time on the cavity surface

---

(*) In reality there is a boundary layer on the body due to the no-slip condition.



boundary layer displacement thickness

$\vec{n}_{BL}$ = normal to boundary layer displaced surface

$$\vec{q}\cdot\vec{n}_{BL}=0 \qquad (50b)$$