Definitions:

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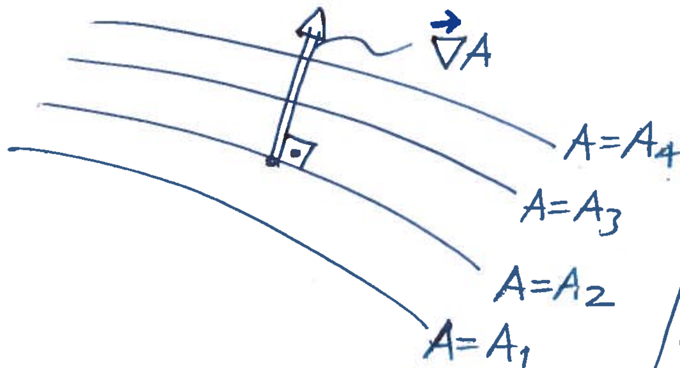
CFD 2/4/02
(revised on 9/11/19)

For A being a scalar (e.g. potential φ, pressure p, temperature T, concentration of contaminant C, etc.)

a) $\vec{\nabla} A \equiv \boldsymbol{\nabla} A \equiv \text{grad}\, A = \vec{i}\,\dfrac{\partial A}{\partial x} + \vec{j}\,\dfrac{\partial A}{\partial y} + \vec{k}\,\dfrac{\partial A}{\partial z}$



surfaces (lines in 2-D) of constant A

b) $\nabla^2 A = \dfrac{\partial^2 A}{\partial x^2} + \dfrac{\partial^2 A}{\partial y^2} + \dfrac{\partial^2 A}{\partial z^2}$ (Laplacian of A)

c) $$\frac{DA}{Dt} \equiv \frac{\partial A}{\partial t} + (\vec{q}\cdot\vec{\nabla})A \equiv \frac{\partial A}{\partial t} + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)A \equiv$$

$$\equiv \frac{\partial A}{\partial t} + u\frac{\partial A}{\partial x} + v\frac{\partial A}{\partial y} + w\frac{\partial A}{\partial z} \equiv \frac{\partial A}{\partial t} + \vec{q}\cdot\vec{\nabla} A$$

(substantial, substantive, material or total derivative with respect to time)

where $\vec{q} = u\vec{i} + v\vec{j} + w\vec{k}$ (velocity vector)

Definitions:

For $\vec{V}$ being a vector (e.g. velocity vector, vorticity vector, etc.):

a) $\vec{\nabla}.\vec{V} \equiv \operatorname{div}\vec{V} \equiv \dfrac{\partial V_x}{\partial x} + \dfrac{\partial V_y}{\partial y} + \dfrac{\partial V_z}{\partial z}$

where: $\vec{V} = V_x\,\vec{i} + V_y\,\vec{j} + V_z\,\vec{k}$

b) $\nabla^2\vec{V} = \nabla^2 V_x\,\vec{i} + \nabla^2 V_y\,\vec{j} + \nabla^2 V_z\,\vec{k}$

c) $\dfrac{D\vec{V}}{Dt} \equiv \dfrac{\partial\vec{V}}{\partial t} + (\vec{q}\cdot\vec{\nabla})\vec{V} \equiv$

$$\frac{DV_x}{Dt}\vec{i} + \frac{DV_y}{Dt}\vec{j} + \frac{DV_z}{Dt}\vec{k} \equiv$$

$$\equiv \left(\frac{\partial V_x}{\partial t} + u\frac{\partial V_x}{\partial x} + v\frac{\partial V_x}{\partial y} + w\frac{\partial V_x}{\partial z}\right)\vec{i} +$$

$$\left(\frac{\partial V_x}{\partial t} + u\frac{V_y}{\partial x} + v\frac{\partial V_y}{\partial y} + w\frac{\partial V_y}{\partial z}\right)\vec{j} +$$

$$\left(\frac{\partial V_z}{\partial t} + u\frac{\partial V_z}{\partial x} + v\frac{\partial V_z}{\partial y} + w\frac{\partial V_z}{\partial z}\right)\vec{k}$$

Where: $\vec{q} = u\vec{i} + v\vec{j} + w\vec{k}$

d) $\vec{\nabla}\times\vec{V} \equiv \operatorname{rot}\vec{V} \equiv \operatorname{curl}\vec{V} \equiv \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \vec{i}\left(\dfrac{\partial V_z}{\partial y} - \dfrac{\partial V_y}{\partial z}\right)$

$$-\vec{j}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right)$$

$$+\vec{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

The vector operator $\vec{\nabla}$:

$$\vec{\nabla} \equiv \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

Examples: a) $\vec{\nabla} A = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)A =$

$$= \vec{i}\frac{\partial A}{\partial x} + \vec{j}\frac{\partial A}{\partial y} + \vec{k}\frac{\partial A}{\partial z}$$

b) $\vec{\nabla}\cdot\vec{V} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(V_x\vec{i} + V_y\vec{j} + V_z\vec{k}\right) =$

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

c) $\vec{q}\cdot\vec{\nabla} = \left(u\vec{i} + v\vec{j} + w\vec{k}\right)\cdot\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) =$

$$= u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

operator which corresponds to the <u>advective</u> or <u>convective</u> terms

d) $\nabla^2 \equiv \vec{\nabla}\cdot\vec{\nabla} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) =$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Some identities:

1) $\vec{\nabla}\times\vec{\nabla}A = 0$ (A: scalar)

2) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{V}) = 0$ ($\vec{V}$: vector)

3) $\vec{\nabla}\cdot(A\vec{V}) = A\,\vec{\nabla}\cdot\vec{V} + \vec{V}\cdot\vec{\nabla}A$

4) $\vec{\nabla}\times(A\vec{V}) = A\,\vec{\nabla}\times\vec{V} + \nabla A\times\vec{V}$

5) $\vec{\nabla}(\vec{u}\cdot\vec{v}) = (\vec{u}\cdot\vec{\nabla})\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{u} + \vec{u}\times(\vec{\nabla}\times\vec{v}) + \vec{v}\times(\vec{\nabla}\times\vec{u})$

6) $\vec{\nabla}\times(\vec{u}\times\vec{v}) = (\vec{v}\cdot\vec{\nabla})\vec{u} - (\vec{u}\cdot\vec{\nabla})\vec{v} + \vec{u}(\vec{\nabla}\cdot\vec{v}) - \vec{v}(\vec{\nabla}\cdot\vec{u})$

7) $\vec{\nabla}\times(\vec{\nabla}\times\vec{q}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{q}) - \nabla^2\vec{q}$

8) $(\vec{q}\cdot\vec{\nabla})\vec{q} = (\vec{\nabla}\times\vec{q})\times\vec{q} + \vec{\nabla}\left(\frac{q^2}{2}\right)$

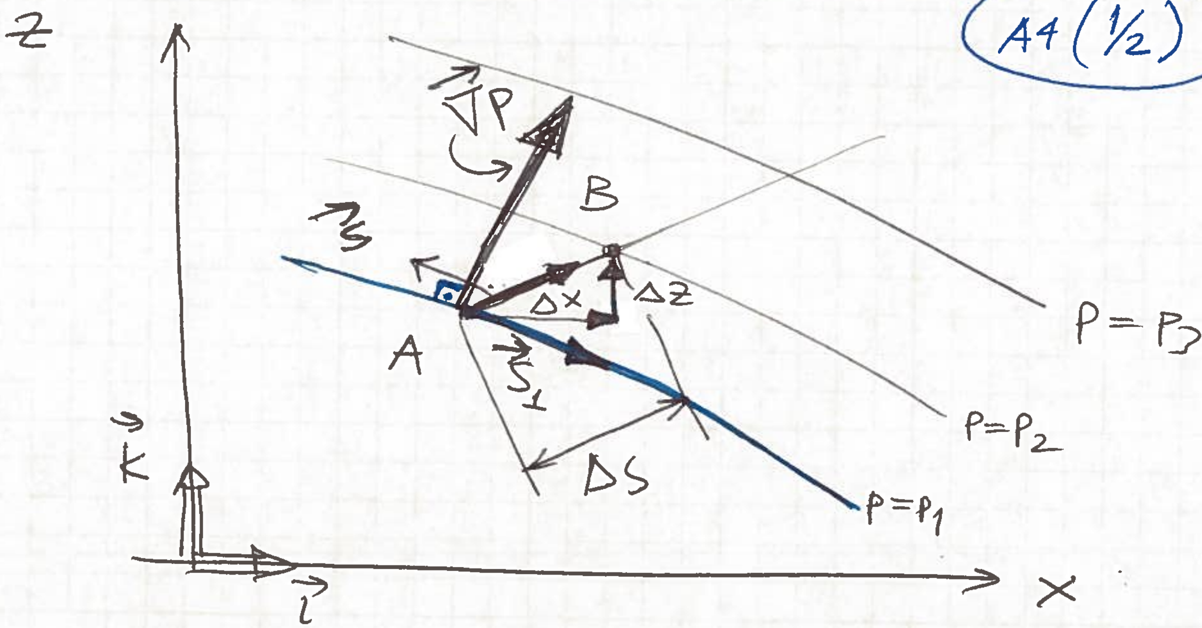
or

$(\vec{q}\cdot\vec{\nabla})\vec{q} = \vec{\omega}\times\vec{q} + \vec{\nabla}\left(\frac{q^2}{2}\right)$

with $\vec{\omega} = \vec{\nabla}\times\vec{q}$ (DEF.)

Physical meaning of gradient

A4 (1/2)



We want to determine $\dfrac{\partial p}{\partial s} \approx \dfrac{\Delta p}{\Delta s}$ (change of p along direction AB)

$$= \frac{p_B - p_A}{\Delta s} = \frac{p(x_B, z_B) - p(x_A, z_A)}{\Delta s} \simeq$$

$$\simeq \frac{\frac{\partial p}{\partial x}\Delta x + \frac{\partial p}{\partial z}\Delta z}{\Delta s} = \underbrace{\left(\frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial z}\vec{k}\right)}_{\substack{\vec{\nabla} p \\ \text{(definition of gradient of } p)}} \cdot \left(\frac{\Delta x}{\Delta s}\vec{i} + \frac{\Delta z}{\Delta s}\vec{k}\right)$$

$$\frac{\Delta x \vec{i} + \Delta z \vec{k}}{\Delta s} = \frac{\vec{AB}}{\Delta s} = \frac{\vec{AB}}{|\vec{AB}|} = \vec{s} \quad (\text{unit vector along } AB)$$

Finally $\boxed{\dfrac{\partial p}{\partial s} = \vec{\nabla} p \cdot \vec{s}}$

Apply above eq. for $\vec{s}_1$ (tangent to $p = p_1$ = constant line) $\Rightarrow \dfrac{\partial p}{\partial s_1} = 0 = \vec{\nabla} p \cdot \vec{s}_1 = 0 \Rightarrow \boxed{\vec{\nabla} p \perp \vec{s}_1}$

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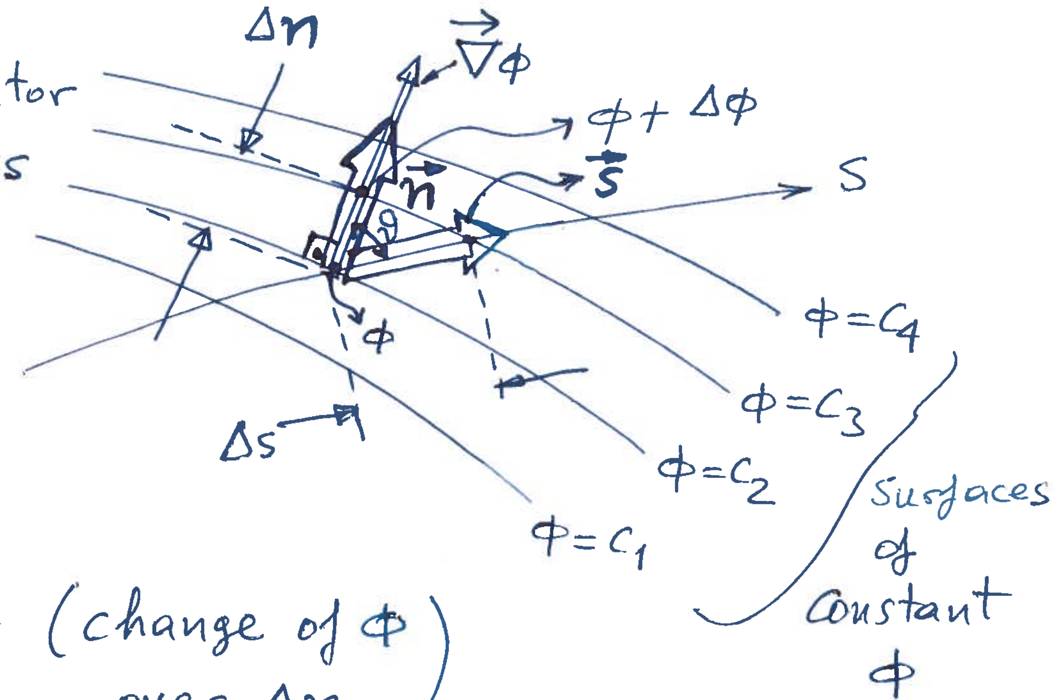
A4 (2/2)

Physical meaning of $grad\,\phi \equiv \vec{\nabla}\phi \equiv \boldsymbol{\nabla}\phi$

$\vec{n}$: unit normal vector

$\vec{s}$: unit vector along s



$$|\vec{\nabla}\phi| = \frac{\Delta\phi}{\Delta n} \quad \left(\text{change of } \phi \text{ over } \Delta n\right)$$

↑ magnitude of vector $\vec{\nabla}\phi$ (assuming $\phi\uparrow$ along n)

Q: How can we express $\dfrac{\Delta\phi}{\Delta s}$ in terms of $\vec{\nabla}\phi$?

A: $$\frac{\Delta\phi}{\Delta s} = \frac{\Delta\phi}{\Delta n}\cdot\frac{\Delta n}{\Delta s} = |\vec{\nabla}\phi|\cdot\cos\theta = |\vec{\nabla}\phi|\cdot|\vec{s}|\cos\theta =$$

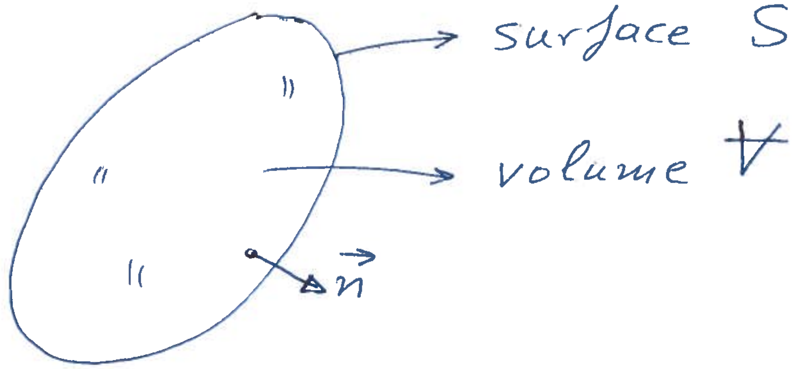
$$= \vec{\nabla}\phi\cdot\vec{s} \quad \text{or} \quad \boxed{\frac{\partial\phi}{\partial s} = \vec{\nabla}\phi\cdot\vec{s}}$$

Special cases:

$\vec{s} = \vec{n} \sim \dfrac{\partial\phi}{\partial n} = \vec{\nabla}\phi\cdot\vec{n} = \pm|\vec{\nabla}\phi|$ $\left(\begin{array}{l} + \text{ if } \phi\uparrow \text{ as } n\uparrow \\ - \text{ if } \phi\downarrow \text{ as } n\uparrow \end{array}\right)$

$\vec{s} \perp \vec{n} \sim \dfrac{\partial\phi}{\partial s} = 0$

Divergence theorem (Gauss's theorem)



$$\underbrace{\iint_S \vec{V}\cdot\vec{n}\, dS}_{\substack{\text{Flux of } \vec{V} \\ \text{through surface } S}} = \iiint_{\forall} \underbrace{\vec{\nabla}\cdot\vec{V}}_{\text{div } \vec{V}}\, d\forall$$

if $\vec{\nabla}\cdot\vec{V} = 0 \Rightarrow \vec{V}$: <u>divergence free</u> vector

or <u>solenoidal</u> vector

<u>Examples:</u>

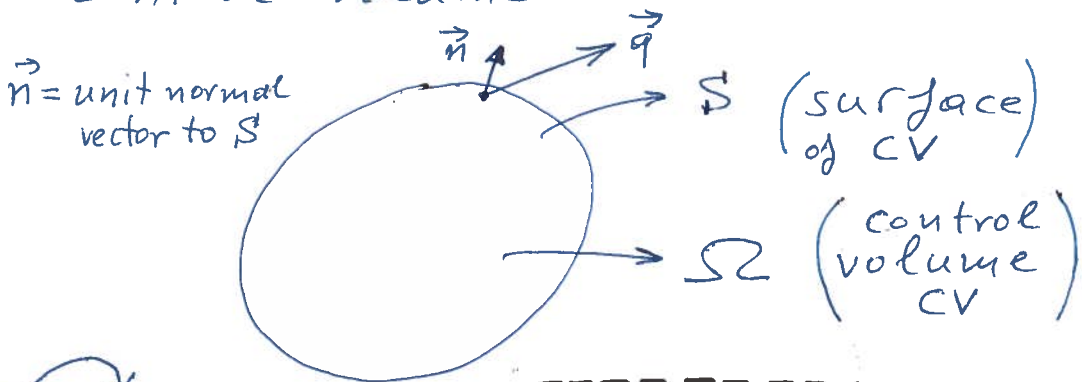
For incompressible flows: $\vec{\nabla}\cdot\vec{q} = 0$

For <u>all</u> flows: $\vec{\nabla}\cdot\vec{\omega} = 0$ (why?)

where $\vec{\omega} = \vec{\nabla}\times\vec{q}$ (vorticity)

Some examples:

Conservation of mass expressed over a control volume:



$$\underbrace{\frac{\partial}{\partial t}\int_{\Omega} \rho \, d\Omega}_{\text{rate of mass production in } \Omega} + \underbrace{\int_{S} \rho \vec{q}\cdot\vec{n} \, dS}_{\text{mass flux through the boundary } S} = 0$$

Using divergence theorem we get

$$\frac{\partial}{\partial t}\int_{\Omega} \rho \, d\Omega + \int_{\Omega} \nabla\cdot(\rho\vec{q}) \, d\Omega = 0 \quad \text{or}$$

$$\int_{\Omega}\left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{q})\right] d\Omega = 0 \;\rightsquigarrow\; \boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{q}) = 0} \quad (1)$$

(since volume integral has to be zero for <u>every</u> Ω)

Using identity (3) on page A3 we express:

$\nabla\cdot(\rho\vec{q}) = \rho\vec{\nabla}\cdot\vec{q} + \vec{q}\cdot\vec{\nabla}\rho$, and (1) becomes:

$$\underbrace{\frac{\partial \rho}{\partial t} + \rho\vec{\nabla}\cdot\vec{q} + \vec{q}\cdot\vec{\nabla}\rho}_{} = 0 \;\rightsquigarrow\; \frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{q} = 0 \;\Bigg\|\; \Rightarrow \boxed{\vec{\nabla}\cdot\vec{q} = 0} \quad (2)$$

For incompressible flows: $\dfrac{D\rho}{Dt} = 0$