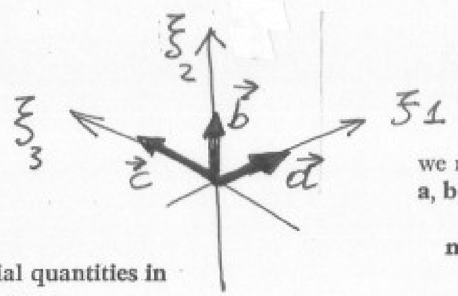


APPENDIX 2

Expressions for some common vector differential quantities in orthogonal curvilinear co-ordinate systems

ξ_1, ξ_2, ξ_3 is a system of orthogonal curvilinear co-ordinates, and the unit vectors **a**, **b**, **c** are parallel to the co-ordinate lines and in the directions of increase of ξ_1, ξ_2, ξ_3 respectively. The change in the position vector **x** corresponding to increments in ξ_1, ξ_2, and ξ_3 can then be written as

$$\delta\mathbf{x} = h_1\,\delta\xi_1\,\mathbf{a} + h_2\,\delta\xi_2\,\mathbf{b} + h_3\,\delta\xi_3\,\mathbf{c}.$$

a, **b**, **c** and the positive scale factors h_1, h_2, h_3 are functions of the co-ordinates.

The fact that the three families of co-ordinate lines form an orthogonal system provides useful expressions for the derivatives of **a**, **b**, and **c**. We have

$$\frac{\partial\mathbf{x}}{\partial\xi_1}\cdot\frac{\partial\mathbf{x}}{\partial\xi_2} = 0,$$

with two other similar relations, and since

$$\frac{\partial}{\partial\xi_3}\left(\frac{\partial\mathbf{x}}{\partial\xi_1}\cdot\frac{\partial\mathbf{x}}{\partial\xi_2}\right) = \frac{\partial}{\partial\xi_1}\left(\frac{\partial\mathbf{x}}{\partial\xi_3}\right)\cdot\frac{\partial\mathbf{x}}{\partial\xi_2} + \frac{\partial\mathbf{x}}{\partial\xi_1}\cdot\frac{\partial}{\partial\xi_2}\left(\frac{\partial\mathbf{x}}{\partial\xi_3}\right)$$

$$= -2\frac{\partial\mathbf{x}}{\partial\xi_3}\cdot\frac{\partial^2\mathbf{x}}{\partial\xi_1\,\partial\xi_2},$$

we see that $\quad \dfrac{\partial^2\mathbf{x}}{\partial\xi_1\,\partial\xi_2}, \quad = \dfrac{\partial(h_2\mathbf{b})}{\partial\xi_1} \quad \text{or} \quad \dfrac{\partial(h_1\mathbf{a})}{\partial\xi_2},$

is a vector normal to **c**. It follows that

$$\frac{\partial\mathbf{a}}{\partial\xi_2} = \frac{1}{h_1}\frac{\partial h_2}{\partial\xi_1}\mathbf{b}, \quad \frac{\partial\mathbf{b}}{\partial\xi_1} = \frac{1}{h_2}\frac{\partial h_1}{\partial\xi_2}\mathbf{a},$$

with four other similar relations. Then

$$\frac{\partial\mathbf{a}}{\partial\xi_1} = \frac{\partial(\mathbf{b}\times\mathbf{c})}{\partial\xi_1} = -\frac{1}{h_2}\frac{\partial h_1}{\partial\xi_2}\mathbf{b} - \frac{1}{h_3}\frac{\partial h_1}{\partial\xi_3}\mathbf{c},$$

with two other similar relations.

The vector *gradient* of a scalar function V is

$$\operatorname{grad} V, \quad \text{or } \nabla V, \quad = \left(\frac{\mathbf{a}}{h_1}\frac{\partial}{\partial\xi_1} + \frac{\mathbf{b}}{h_2}\frac{\partial}{\partial\xi_2} + \frac{\mathbf{c}}{h_3}\frac{\partial}{\partial\xi_3}\right)V.$$

The gradient in a direction **n** is obtained from the operator $\mathbf{n}.\nabla$, which may act on either a scalar or a vector. To find the components of $\mathbf{n}.\nabla\mathbf{F}$, where

$$\mathbf{F} = F_1\mathbf{a} + F_2\mathbf{b} + F_3\mathbf{c},$$

we must allow for the dependence of both F_1, F_2, F_3 and the unit vectors **a**, **b**, **c** on position. It follows from the above relations that

$$\mathbf{n}.\nabla\mathbf{F} = \mathbf{a}\left\{\mathbf{n}.\nabla F_1 + \frac{F_2}{h_1 h_2}\left(n_1\frac{\partial h_1}{\partial\xi_2} - n_2\frac{\partial h_2}{\partial\xi_1}\right) + \frac{F_3}{h_3 h_1}\left(n_1\frac{\partial h_1}{\partial\xi_3} - n_3\frac{\partial h_3}{\partial\xi_1}\right)\right\}$$

$$+\mathbf{b}\{\quad\} + \mathbf{c}\{\quad\},$$

where n_1, n_2, n_3 are the components of **n** in the directions **a**, **b**, **c**.

The *divergence* and *curl* operators act only on a vector, and

$$\operatorname{div}\mathbf{F}, \quad \text{or } \nabla.\mathbf{F}, \quad = \frac{\mathbf{a}}{h_1}\cdot\frac{\partial\mathbf{F}}{\partial\xi_1} + \frac{\mathbf{b}}{h_2}\cdot\frac{\partial\mathbf{F}}{\partial\xi_2} + \frac{\mathbf{c}}{h_3}\cdot\frac{\partial\mathbf{F}}{\partial\xi_3},$$

$$\operatorname{curl}\mathbf{F}, \quad \text{or } \nabla\times\mathbf{F}, \quad = \frac{\mathbf{a}}{h_1}\times\frac{\partial\mathbf{F}}{\partial\xi_1} + \frac{\mathbf{b}}{h_2}\times\frac{\partial\mathbf{F}}{\partial\xi_2} + \frac{\mathbf{c}}{h_3}\times\frac{\partial\mathbf{F}}{\partial\xi_3}.$$

By making use of the expressions for derivatives of **a**, **b** and **c**, we find

$$\nabla.\mathbf{F} = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial(h_2 h_3 F_1)}{\partial\xi_1} + \frac{\partial(h_3 h_1 F_2)}{\partial\xi_2} + \frac{\partial(h_1 h_2 F_3)}{\partial\xi_3}\right\};$$

this can also be regarded as the result of applying the 'divergence theorem' to the small parallelepiped whose edges are displacements along co-ordinate lines corresponding to the increments $\delta\xi_1, \delta\xi_2, \delta\xi_3$. Likewise we find

$$\nabla\times\mathbf{F} = \frac{\mathbf{a}}{h_2 h_3}\left\{\frac{\partial(h_3 F_3)}{\partial\xi_2} - \frac{\partial(h_2 F_2)}{\partial\xi_3}\right\} + \frac{\mathbf{b}}{h_3 h_1}\left\{\frac{\partial(h_1 F_1)}{\partial\xi_3} - \frac{\partial(h_3 F_3)}{\partial\xi_1}\right\}$$

$$+ \frac{\mathbf{c}}{h_1 h_2}\left\{\frac{\partial(h_2 F_2)}{\partial\xi_1} - \frac{\partial(h_1 F_1)}{\partial\xi_2}\right\},$$

$$\text{or} \quad \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\mathbf{a} & h_2\mathbf{b} & h_3\mathbf{c} \\ \dfrac{\partial}{\partial\xi_1} & \dfrac{\partial}{\partial\xi_2} & \dfrac{\partial}{\partial\xi_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix},$$

which can also be regarded as following from the application of Stokes's theorem in turn to three orthogonal faces of the same parallelepiped.

The divergence of the gradient gives the *Laplacian* operator, which may act on either a scalar or a vector.

$$\nabla.\nabla V, \quad \text{or } \nabla^2 V, \quad = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial\xi_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial\xi_1}\right)\right.$$

$$\left. + \frac{\partial}{\partial\xi_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial V}{\partial\xi_2}\right) + \frac{\partial}{\partial\xi_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial V}{\partial\xi_3}\right)\right\}.$$

The components of $\nabla^2\mathbf{F}$ may be calculated by replacing V in this formula by $\mathbf{F}, = F_1\mathbf{a} + F_2\mathbf{b} + F_3\mathbf{c}$, and using the expressions for derivatives of **a**, **b** and **c**, but the result is too complicated to be useful. It is usually more convenient,

From "An Introduction to Fluid Dynamics"
G. K. Batchelor, Cambridge Univ. Press

when finding the components of $\nabla^2\mathbf{F}$ in a particular co-ordinate system, to use the identity

$$\nabla^2\mathbf{F} = \nabla(\nabla . \mathbf{F}) - \nabla\times(\nabla\times\mathbf{F})$$

and the above expressions for grad, div and curl.

Consider now the components of the rate-of-strain tensor expressed in terms of velocity components and derivatives relative to the curvilinear system. The gradient, in the direction $\mathbf{n}$, of the component of velocity $\mathbf{u}$ in the fixed direction $\mathbf{m}$ is

$$\mathbf{n}.\nabla(\mathbf{m}.\mathbf{u}), \quad = \mathbf{m}.(\mathbf{n}.\nabla\mathbf{u}).$$

Diagonal elements of the rate-of-strain tensor represent rates of extension, obtained by putting $\mathbf{m} = \mathbf{n}$, and the non-diagonal elements involve velocity gradients for which $\mathbf{m}$ and $\mathbf{n}$ are orthogonal. We see then, from the above formula for $\mathbf{n}.\nabla\mathbf{F}$, that the components of the rate-of-strain tensor relative to Cartesian axes locally parallel to $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ (to which the suffixes 1, 2, 3 refer, respectively) are

$$e_{11} = \mathbf{a}.(\mathbf{a}.\nabla\mathbf{u}) = \frac{1}{h_1}\frac{\partial u_1}{\partial \xi_1} + \frac{u_2}{h_1 h_2}\frac{\partial h_1}{\partial \xi_2} + \frac{u_3}{h_3 h_1}\frac{\partial h_1}{\partial \xi_3},$$

$$e_{23} = \tfrac{1}{2}\mathbf{b}.(\mathbf{c}.\nabla\mathbf{u}) + \tfrac{1}{2}\mathbf{c}.(\mathbf{b}.\nabla\mathbf{u}) = \frac{h_3}{2h_2}\frac{\partial}{\partial \xi_2}\left(\frac{u_3}{h_3}\right) + \frac{h_2}{2h_3}\frac{\partial}{\partial \xi_3}\left(\frac{u_2}{h_2}\right),$$

with four other expressions obtained by cyclic interchange of suffixes. The components of the stress tensor σ_{ij} can be obtained from those of rate of strain, using the relation (for an incompressible fluid)

$$\sigma_{ij} = -p\,\delta_{ij} + 2\mu e_{ij}.$$

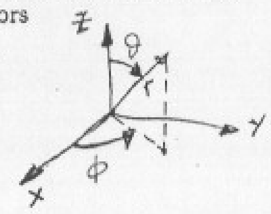
The components of all terms in the equation of motion of a fluid in the directions $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ may now be found by simple substitution in the appropriate expressions above. The components of the term $\mathbf{u}.\nabla\mathbf{u}$ in the acceleration are obtained from the expression for $\mathbf{n}.\nabla\mathbf{F}$.

Applications to some particular co-ordinate systems are as follows.

Spherical polar co-ordinates

To the co-ordinates $\xi_1 = r$, $\xi_2 = \theta$, $\xi_3 = \phi$ (where ϕ is the azimuthal angle about the axis $\theta = 0$) there correspond the scale factors

$$h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta.$$



Then

$$\frac{\partial\mathbf{a}}{\partial r} = 0, \quad \frac{\partial\mathbf{a}}{\partial\theta} = \mathbf{b}, \quad \frac{\partial\mathbf{a}}{\partial\phi} = \sin\theta\,\mathbf{c},$$

$$\frac{\partial\mathbf{b}}{\partial r} = 0, \quad \frac{\partial\mathbf{b}}{\partial\theta} = -\mathbf{a}, \quad \frac{\partial\mathbf{b}}{\partial\phi} = \cos\theta\,\mathbf{c},$$

$$\frac{\partial\mathbf{c}}{\partial r} = 0, \quad \frac{\partial\mathbf{c}}{\partial\theta} = 0, \quad \frac{\partial\mathbf{c}}{\partial\phi} = -\sin\theta\,\mathbf{a} - \cos\theta\,\mathbf{b}.$$

$$\nabla V = \mathbf{a}\frac{\partial V}{\partial r} + \frac{\mathbf{b}}{r}\frac{\partial V}{\partial\theta} + \frac{\mathbf{c}}{r\sin\theta}\frac{\partial V}{\partial\phi},$$

$$\mathbf{n}.\nabla\mathbf{F} = \mathbf{a}\left(\mathbf{n}.\nabla F_r - \frac{n_\theta F_\theta}{r} - \frac{n_\phi F_\phi}{r}\right) + \mathbf{b}\left(\mathbf{n}.\nabla F_\theta - \frac{n_\phi F_\phi}{r}\cot\theta + \frac{n_\theta F_r}{r}\right)$$
$$+ \mathbf{c}\left(\mathbf{n}.\nabla F_\phi + \frac{n_\phi F_r}{r} + \frac{n_\phi F_\theta}{r}\cot\theta\right),$$

$$\nabla.\mathbf{F} = \frac{1}{r^2}\frac{\partial(r^2 F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,F_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi},$$

$$\nabla\times\mathbf{F} = \frac{\mathbf{a}}{r\sin\theta}\left\{\frac{\partial(F_\phi\sin\theta)}{\partial\theta} - \frac{\partial F_\theta}{\partial\phi}\right\} + \frac{\mathbf{b}}{r}\left\{\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial(rF_\phi)}{\partial r}\right\} + \frac{\mathbf{c}}{r}\left\{\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right\},$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2},$$

$$\nabla^2\mathbf{F} = \mathbf{a}\left\{\nabla^2 F_r - \frac{2F_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(F_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial F_\phi}{\partial\phi}\right\}$$
$$+ \mathbf{b}\left\{\nabla^2 F_\theta + \frac{2}{r^2}\frac{\partial F_r}{\partial\theta} - \frac{F_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial F_\phi}{\partial\phi}\right\}$$
$$+ \mathbf{c}\left\{\nabla^2 F_\phi + \frac{2}{r^2\sin\theta}\frac{\partial F_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial F_\theta}{\partial\phi} - \frac{F_\phi}{r^2\sin^2\theta}\right\}.$$

Rate-of-strain tensor:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}, \quad e_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r},$$

$$e_{\theta\phi} = \frac{\sin\theta}{2r}\frac{\partial}{\partial\theta}\left(\frac{u_\phi}{\sin\theta}\right) + \frac{1}{2r\sin\theta}\frac{\partial u_\theta}{\partial\phi}, \quad e_{\phi r} = \frac{1}{2r\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right),$$

$$e_{r\theta} = \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{2r}\frac{\partial u_r}{\partial\theta}.$$

Equation of motion for an incompressible fluid, with no body force:

$$\frac{\partial u_r}{\partial t} + \mathbf{u}.\nabla u_r - \frac{u_\theta^2}{r} - \frac{u_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$
$$+ \nu\left\{\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right\},$$

$$\frac{\partial u_\theta}{\partial t} + \mathbf{u}.\nabla u_\theta + \frac{u_r u_\theta}{r} - \frac{u_\phi^2\cot\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta}$$
$$+ \nu\left\{\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}\right\},$$

$$\frac{\partial u_\phi}{\partial t} + \mathbf{u}.\nabla u_\phi + \frac{u_\phi u_r}{r} + \frac{u_\theta u_\phi\cot\theta}{r} = -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\phi}$$
$$+ \nu\left\{\nabla^2 u_\phi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi} - \frac{u_\phi}{r^2\sin^2\theta}\right\}.$$

Cylindrical co-ordinates

To the co-ordinates $\xi_1 = x$, $\xi_2 = \sigma$, $\xi_3 = \phi$ (where ϕ is the azimuthal angle about the axis $\sigma = 0$) there correspond the scale factors

$$h_1 = 1, \quad h_2 = 1, \quad h_3 = \sigma.$$

Then

$$\frac{\partial \mathbf{a}}{\partial \phi} = 0, \quad \frac{\partial \mathbf{b}}{\partial \phi} = \mathbf{c}, \quad \frac{\partial \mathbf{c}}{\partial \phi} = -\mathbf{b},$$

and **a**, **b**, **c** are independent of x and σ.

$$\nabla V = \mathbf{a}\frac{\partial V}{\partial x} + \mathbf{b}\frac{\partial V}{\partial \sigma} + \frac{\mathbf{c}}{\sigma}\frac{\partial V}{\partial \phi},$$

$$\mathbf{n}.\nabla \mathbf{F} = \mathbf{a}(\mathbf{n}.\nabla F_x) + \mathbf{b}\left(\mathbf{n}.\nabla F_\sigma - \frac{n_\phi F_\phi}{\sigma}\right) + \mathbf{c}\left(\mathbf{n}.\nabla F_\phi + \frac{n_\phi F_\sigma}{\sigma}\right),$$

$$\nabla.\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{1}{\sigma}\frac{\partial(\sigma F_\sigma)}{\partial \sigma} + \frac{1}{\sigma}\frac{\partial F_\phi}{\partial \phi},$$

$$\nabla \times \mathbf{F} = \mathbf{a}\left\{\frac{1}{\sigma}\frac{\partial(\sigma F_\phi)}{\partial \sigma} - \frac{1}{\sigma}\frac{\partial F_\sigma}{\partial \phi}\right\} + \mathbf{b}\left(\frac{1}{\sigma}\frac{\partial F_x}{\partial \phi} - \frac{\partial F_\phi}{\partial x}\right) + \mathbf{c}\left(\frac{\partial F_\sigma}{\partial x} - \frac{\partial F_x}{\partial \sigma}\right),$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{1}{\sigma}\frac{\partial}{\partial \sigma}\left(\sigma\frac{\partial V}{\partial \sigma}\right) + \frac{1}{\sigma^2}\frac{\partial^2 V}{\partial \phi^2},$$

$$\nabla^2 \mathbf{F} = \mathbf{a}(\nabla^2 F_x) + \mathbf{b}\left(\nabla^2 F_\sigma - \frac{F_\sigma}{\sigma^2} - \frac{2}{\sigma^2}\frac{\partial F_\phi}{\partial \phi}\right) + \mathbf{c}\left(\nabla^2 F_\phi + \frac{2}{\sigma^2}\frac{\partial F_\sigma}{\partial \phi} - \frac{F_\phi}{\sigma^2}\right).$$

Rate-of-strain tensor:

$$e_{xx} = \frac{\partial u_x}{\partial x}, \quad e_{\sigma\sigma} = \frac{\partial u_\sigma}{\partial \sigma}, \quad e_{\phi\phi} = \frac{1}{\sigma}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\sigma}{\sigma},$$

$$e_{\sigma\phi} = \frac{\sigma}{2}\frac{\partial}{\partial \sigma}\left(\frac{u_\phi}{\sigma}\right) + \frac{1}{2\sigma}\frac{\partial u_\sigma}{\partial \phi}, \quad e_{\phi x} = \frac{1}{2\sigma}\frac{\partial u_x}{\partial \phi} + \frac{1}{2}\frac{\partial u_\phi}{\partial x}, \quad e_{x\sigma} = \frac{1}{2}\frac{\partial u_\sigma}{\partial x} + \frac{1}{2}\frac{\partial u_x}{\partial \sigma}.$$

Equation of motion for an incompressible fluid, with no body force:

$$\frac{\partial u_x}{\partial t} + \mathbf{u}.\nabla u_x = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u_x,$$

$$\frac{\partial u_\sigma}{\partial t} + \mathbf{u}.\nabla u_\sigma - \frac{u_\phi^2}{\sigma} = -\frac{1}{\rho}\frac{\partial p}{\partial \sigma} + \nu\left(\nabla^2 u_\sigma - \frac{u_\sigma}{\sigma^2} - \frac{2}{\sigma^2}\frac{\partial u_\phi}{\partial \phi}\right),$$

$$\frac{\partial u_\phi}{\partial t} + \mathbf{u}.\nabla u_\phi + \frac{u_\sigma u_\phi}{\sigma} = -\frac{1}{\rho\sigma}\frac{\partial p}{\partial \phi} + \nu\left(\nabla^2 u_\phi + \frac{2}{\sigma^2}\frac{\partial u_\sigma}{\partial \phi} - \frac{u_\phi}{\sigma^2}\right).$$

Polar co-ordinates in two dimensions

The relevant formulae can be obtained from those for the above cylindrical co-ordinates by suppressing all components and derivatives in the direction

of the x-co-ordinate line, but are written out here in view of the frequency of their use. The co-ordinates are

$$\xi_1 = r, \quad \xi_2 = \theta, \quad \text{and} \quad h_1 = 1, \quad h_2 = r,$$

$$\frac{\partial \mathbf{a}}{\partial r} = 0, \quad \frac{\partial \mathbf{a}}{\partial \theta} = \mathbf{b}, \quad \frac{\partial \mathbf{b}}{\partial r} = 0, \quad \frac{\partial \mathbf{b}}{\partial \theta} = -\mathbf{a}.$$

$$\nabla V = \mathbf{a}\frac{\partial V}{\partial r} + \frac{\mathbf{b}}{r}\frac{\partial V}{\partial \theta},$$

$$\mathbf{n}.\nabla \mathbf{F} = \mathbf{a}\left(\mathbf{n}.\nabla F_r - \frac{n_\theta F_\theta}{r}\right) + \mathbf{b}\left(\mathbf{n}.\nabla F_\theta + \frac{n_\theta F_r}{r}\right),$$

$$\nabla.\mathbf{F} = \frac{1}{r}\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta},$$

$$\nabla \times \mathbf{F} = \left\{\frac{1}{r}\frac{\partial(rF_\theta)}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial \theta}\right\}\mathbf{a}\times\mathbf{b},$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2},$$

$$\nabla^2 \mathbf{F} = \mathbf{a}\left(\nabla^2 F_r - \frac{F_r}{r^2} - \frac{2}{r^2}\frac{\partial F_\theta}{\partial \theta}\right) + \mathbf{b}\left(\nabla^2 F_\theta + \frac{2}{r^2}\frac{\partial F_r}{\partial \theta} - \frac{F_\theta}{r^2}\right).$$

Rate-of-strain tensor:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad e_{r\theta} = \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{2r}\frac{\partial u_r}{\partial \theta}.$$

Equation of motion for an incompressible fluid, with no body force:

$$\frac{\partial u_r}{\partial t} + \left(u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta}\right)u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right),$$

$$\frac{\partial u_\theta}{\partial t} + \left(u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta}\right)u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right).$$